

Deformation quantization on jet manifolds

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Abstract

Deformation quantization conventionally is described in terms of multidifferential operators. Jet manifold technique is well-known provide the adequate formulation of theory of differential operators. We extended this formulation to the multidifferential ones, and consider their infinite order jet prolongation. The infinite order jet manifold is endowed with the canonical flat connection that provides the covariant formula of a deformation star-product.

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1 Introduction

Deformation quantization conventionally is described in terms of multidifferential operators. Jet formalism provides the adequate formulation of theory of differential operators and differential equations on manifolds and fibre bundles [5, 19, 31]. Therefore, we develop an idea in [6, 7, 20] and aim to describe deformation quantization in terms of jets.

For this purpose, we develop theory of differential and multidifferential operators on a ring $C^\infty(X)$ of smooth real functions on a smooth manifold X as functions on an infinite order jet manifold $J^\infty F$ (3.14) (Definition 3.3 and Theorem 3.2). Their Hochschild complex (3.28) is constructed, and the deformation (3.59) is defined. A key point is that an infinite order jet manifold $J^\infty F$ is endowed with the canonical flat connection (3.104) that provides the covariant formula (3.114) of a deformation star-product.

2 Deformation quantization of Poisson manifolds

In a general setting, a deformation of an algebra A over a commutative ring \mathcal{K} is its Gerstenhaber extension to an algebra \mathcal{A}_h (Definition 2.1) over the ring $\mathcal{K}[[h]]$ of formal power series in a real number $h > 0$, called the deformation parameter [17]. In these terms, deformation quantization is defined to be a deformation of a Poisson algebra of smooth real functions on a Poisson manifold where h is treated as a Plank constant [2].

2.1 Gerstenhaber's deformation of algebras

This Section summarizes the relevant material on deformations of algebraic structures [14, 17, 18, 20].

2.1.1 Formal deformation

Let A be a (not necessarily associative) algebra. Let us consider a set $A[[h]]$ of formal power series

$$a_h = a + ha_1 + h^2a_2 + \cdots, \quad a_i \in A,$$

in a deformation parameter h whose coefficients are elements of A . It naturally is an algebra, called the power series algebra, with respect to the formal sum and product of power series.

Let \mathcal{K} be a commutative ring (i.e. a unital associative algebra), and let A be a \mathcal{K} -algebra. Then $\mathcal{K}[[h]]$ also is a commutative ring, and a power series algebra $A[[h]]$ is both a $\mathcal{K}[[h]]$ -module and a $\mathcal{K}[[h]]$ -algebra. It also is a \mathcal{K} -algebra, and there is a canonical monomorphism $A \rightarrow A[[h]]$.

A power series algebra $A[[h]]$ can be provided with a different algebraic structure as follows. Let α_r , $r = 1, \dots$, be a series of \mathcal{K} -bilinear maps $A \times A \rightarrow A$

extended to $A[[h]]$. Let us define a $\mathcal{K}[[h]]$ bilinear map multiplication

$$a * b = \alpha_h(a, b) = ab + \sum_{r=1} h^r \alpha_r(a, b), \quad a, b \in A[[h]], \quad (2.1)$$

in the $\mathcal{K}[[h]]$ -module $A[[h]]$.

DEFINITION 2.1: The multiplication (2.1) is called the formal deformation (or, in short, a deformation) of an original multiplication in A , and makes $A[[h]]$ into a $\mathcal{K}[[h]]$ -algebra \mathcal{A}_h called the (formal) deformation of A . \square

One can think of the multiplication (2.1) as being the power series

$$\alpha_h = \alpha + \sum_{r=1} h^r \alpha_r,$$

of the multiplications α_r in $A[[h]]$

Remark 2.1: For instance, the power series algebra $A[[h]]$ with the original multiplication law $a * b = ab$ is called the null deformation of A . \square

Remark 2.2: A \mathcal{K} -algebra \mathcal{A}_h contains a two-sided ideal generated by elements of $hA \subset \mathcal{A}_h$ so that $\mathcal{A}_h/h\mathcal{A}_h = A$ \square

DEFINITION 2.2: Two deformations \mathcal{A}_h and \mathcal{A}'_h of a \mathcal{K} -algebra A are said to be equivalent if they are isomorphic $\mathcal{K}[[h]]$ -algebras, i.e., there exists an isomorphism of $\mathcal{K}[[h]]$ -modules

$$\phi_h : A[[h]] \rightarrow A[[h]] \quad (2.2)$$

such that the relation

$$\phi_h(\alpha'_h(a, b)) = \alpha_h(\phi_h(a), \phi_h(b)) \quad (2.3)$$

holds. \square

For the sake of brevity, let us write $\alpha'_h = \alpha_h \circ \phi_h$. Any $\mathcal{K}[[h]]$ -linear morphism (2.2) is necessarily a formal power series

$$\phi_h = \phi + h\phi_1 + h^2\phi_2 + \cdots, \quad (2.4)$$

whose coefficients are \mathcal{K} -linear maps $\phi_i : A \rightarrow A$. Substituting this power series into the relation (2.3), one easily obtains that

$$\phi = \text{Id } A, \quad \phi_1(ab) + \alpha'_1(a, b) = \alpha_1(a, b) + \phi_1(a)b + a\phi_1(b). \quad (2.5)$$

DEFINITION 2.3: A deformation \mathcal{A}_h of A is said to be trivial if it is equivalent to the null deformation $A[[h]]$ of A . \square

Remark 2.3: A $\mathcal{K}[[h]]$ -algebra automorphism ϕ_h of the power series algebra $A[[h]]$ obeys the relation

$$\phi_h(ab) = \phi_h(a)\phi_h(b), \quad a, b \in A. \quad (2.6)$$

In particular, the second equality (2.5) takes a form

$$\phi_1(ab) = \phi_1(a)b + a\phi_1(b),$$

i.e., ϕ_1 is a derivation of an algebra A . \square

DEFINITION 2.4: Let $\mathcal{S} \rightarrow A$ be an algebra morphism. A deformation \mathcal{A}_h of A is said to be the \mathcal{S} -relative deformation if

$$a * s = as, \quad s * a = sa$$

for all $a \in A$ and $s \in \mathcal{S}$. \square

For instance, let A be a unital algebra and $\mathcal{S} = \{1\}$. One can show that any deformation of A is unital.

2.1.2 Deformation of rings

Let A be \mathcal{K} -ring, i.e., a unital associative algebra over a commutative ring \mathcal{K} . We aim to study its deformations (2.1) which are rings. It is readily observed that the null deformation of a ring is a ring. A ring which admits only trivial deformations is called rigid.

LEMMA 2.1: One can show the following.

- (i) Any deformation of A is equivalent to that where the unit element coincides with the unit element of A .
- (ii) Invertible elements of A remain invertible in any deformation. \square

In order to be associative, a deformation \mathcal{A}_h of A must satisfy the associativity condition

$$\begin{aligned} (a_1 * a_2) * a_3 - a_1 * (a_2 * a_3) &= \sum_{k=1} h^k D_k(a_1, a_2, a_3) = 0, \\ D_k(a_1, a_2, a_3) &= \sum_{s+r=k, s, r \geq 0} [\alpha_r(\alpha_s(a_1, a_2), a_3) - \alpha_r(a_1, \alpha_s(a_2, a_3))], \end{aligned} \quad (2.7)$$

that is,

$$D_k(a_1, a_2, a_3) = 0, \quad k = 1, \dots, \quad a_i \in A. \quad (2.8)$$

This condition is phrased in terms of the Hochschild cohomology (Definition 4.3) as follows.

Let $E_k(a_1, a_2, a_3)$ denote the sum of the terms with indices $1 \leq s, r \leq k$ in the right-hand side of the expression (2.7). It is easily observed that $E_1 = 0$, while each E_k , $k = 2, \dots$, depends on the terms α_i , $i = 1, \dots, k-1$. Then the condition (2.8) takes a form

$$D_k(a_1, a_2, a_3) = E_k(a_1, a_2, a_3) - (\delta\alpha_k)(a_1, a_2, a_3) = 0, \quad (2.9)$$

where

$$\begin{aligned} (\delta\alpha_k)(a_1, a_2, a_3) &= a_1\alpha_k(a_2, a_3) - \alpha_k(a_1a_2, a_3) + \\ &\quad \alpha_k(a_1, a_2a_3) - \alpha_k(a_1, a_2)a_3 \end{aligned} \quad (2.10)$$

is the Hochschild coboundary operator (4.64) of the Hochschild complex $B^*(A, A)$ (4.60). Thus, one can think of the terms α_k of the power series (2.1) as being two-cochains of the above mentioned Hochschild complex $B^*(A, A)$. Since $E_1 = 0$, a glance at the condition (2.9) shows that α_1 is a two-cocycle of the Hochschild complex. Then we can obtain an associative deformation in the framework of the following recurrence procedure. Let us assume that $\alpha_i, i \leq k$, are two-cochains such that $D_i = 0$ for all $i \leq k$. One can show that $\delta E_{k+1} = 0$, i.e., E_{k+1} is a three-cocycle whose cohomology class depends only on that of $\alpha_i, i = 1, \dots, k$. If this cocycle is a coboundary, i.e., it belongs to the zero element of the Hochschild cohomology group $H^3(A, A)$, then a desired cochain α_{k+1} can be found.

For instance, let \mathcal{A}_h be a deformation of A such that $\alpha_i = 0, i = 1, \dots, k-1, k > 1$. Then $E_k = 0$, and the condition (2.9) shows that α_k is a two-cocycle. As was mentioned above, this also is the case of $k = 1$. The Hochschild cohomology class $[\alpha_k]$ of α_k (or, if there is no danger of confusion, α_k itself) is called the infinitesimal of a deformation. Let us assume that $\alpha_k = \delta\phi$ is a coboundary. It is easily verified that

$$\alpha'_h = \alpha_h \circ (\text{Id } A - h^k \phi) \quad (2.11)$$

is an equivalent deformation such that $\alpha'_i = 0, i = 1, \dots, k$. If $H^2(A, A) = 0$ and α_h is a deformation, one can use the equivalences (2.11) in order to remove all the terms α_i as follows.

THEOREM 2.2: Equivalent deformations of A possess the same infinitesimal $[\alpha_1] \in H^2(A, A)$. \square

THEOREM 2.3: If A is a \mathcal{K} -ring with $H^2(A, A) = 0$, then any deformation of A is trivial, i.e., A is rigid. \square

A ring A is called absolutely rigid if $H^2(A, A) = 0$.

Let $H^3(A, A) = 0$. For any two-cocycle α , there exists a deformation of A such that $\alpha_1 = \alpha$. Indeed, $\alpha_1 = \alpha$ defines E_2 by the formulae (2.7) and (2.9). This three-cocycle is a coboundary. Therefore, there exists a two-cochain α_2 such that the term D_2 (2.9) vanishes. The two-cochains α_1 and α_2 define the three-cocycle E_3 by the formulae (2.7) and (2.9). It also is a coboundary. Consequently, there exists a two-cochain α_3 such that the term D_3 (2.9) vanishes, and so on. Thus, elements of the Hochschild cohomology group $H^3(A, A)$ provide the obstruction to a Hochschild two-cocycle be the infinitesimal of a deformation.

Example 2.4: Let \mathcal{K} be a \mathbb{Q} -ring. Let u and v be derivations of a \mathcal{K} -ring A . They are Hochschild one-cocycles. Their cup-product $u \smile v$ (4.65) is a two-cocycle. This two-cocycle need not be the infinitesimal of a deformation of

A , unless u and v mutually commute. If the derivations u and v commute, they define a deformation of A given by the formula

$$a * b = \exp(hu \smile v)(a, b) = ab + \sum_{r=1} \frac{h^r}{r!} u^r(a) v^r(b), \quad a, b \in A.$$

□

2.2 Star-product

A Poisson bracket on the ring $C^\infty(Z)$ of smooth real functions on a manifold Z (or a Poisson structure on Z) is defined as an \mathbb{R} -bilinear map

$$C^\infty(Z) \times C^\infty(Z) \ni (f, g) \rightarrow \{f, g\} \in C^\infty(Z)$$

which satisfies the following conditions:

- $\{g, f\} = -\{f, g\}$ (skew-symmetry);
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (the Jacobi identity);
- $\{h, fg\} = \{h, f\}g + f\{h, g\}$ (the Leibniz rule).

A manifold Z endowed with a Poisson structure is called a Poisson manifold. A Poisson bracket makes $C^\infty(Z)$ into a real Lie algebra, called the Poisson algebra. A Poisson structure is characterized by a particular bivector field as follows.

THEOREM 2.4: Every Poisson bracket on a manifold Z is uniquely defined as

$$\{f, f'\} = w(df, df') = w^{\mu\nu} \partial_\mu f \partial_\nu f' \quad (2.12)$$

by the bivector field w (4.22) whose Schouten–Nijenhuis bracket $[w, w]_{\text{SN}}$ (4.24) vanishes. It is called a Poisson bivector field [51]. □

Let $(Z, \{, \})$ be a Poisson manifold. By a star-product on Z is meant a ring deformation

$$f * f' = ff' + \sum_{r=1} h^r \alpha_r(f, f') \quad (2.13)$$

of a real ring $C^\infty(Z)$ of smooth real functions on Z such that

$$\alpha_1(f, f') - \alpha_1(f', f) = 2\{f, f'\}, \quad f, f' \in C^\infty(Z). \quad (2.14)$$

In accordance with Lemma 2.1, one can always choose $*$ such that

$$f * 1 = 1 * f = f, \quad f \in C^\infty(Z).$$

Remark 2.5: Given the star-product (2.13) – (2.14), the commutator

$$\begin{aligned} [f, f']_h &= (2h)^{-1}(f * f' - f' * f) = \{f, f'\} + \\ &\quad \frac{1}{2} \sum_{r=1} h^r (\alpha_{r+1}(f, f') - \alpha_{r+1}(f', f)) \end{aligned} \quad (2.15)$$

provides a Lie deformation of the Poisson bracket $\{\cdot, \cdot\}$ on Z . This deformation is treated as deformation quantization. \square

Remark 2.6: Here, we are not concerned with the Laurent series $\mathcal{K}[h^{-1}, h]$ in h , i.e., a polynomial in h^{-1} and a formal series in h . \square

Hereafter, we restrict our consideration to differential deformations (2.13) where $\alpha_r(f, f')$ are bilinear differential (bidifferential) operators of finite order. They are equivalent to continuous deformations as follows (Theorem 2.5).

Remark 2.7: Whenever referring to a topology on the ring $C^\infty(X)$, we will mean the topology of compact convergence for all derivatives [45]. The $C^\infty(X)$ is a Fréchet ring with respect to this topology, i.e., a complete metrizable locally convex topological vector space. \square

Because $C^\infty(Z)$ is a Fréchet ring (Remark 2.7), by its continuous deformation is meant the deformation (2.13) where bilinear maps α_r are continuous in each argument. Continuous cochains form a subcomplex $B_c^*(C^\infty(Z), C^\infty(Z))$ of the Hochschild complex $B^*(C^\infty(Z), C^\infty(Z))$ (4.60). Let $H_c^*(C^\infty(Z), C^\infty(Z))$ denote its cohomology, called the continuous Hochschild cohomology. Another subcomplex $B_d^*(C^\infty(Z), C^\infty(Z))$ of the Hochschild complex $B^*(C^\infty(Z), C^\infty(Z))$ consists of cochains which are $C^\infty(X)$ -valued multidifferential operators on $C^\infty(X)$ of finite order. Let us denote

$$\mathcal{D}^* = B^*(C^\infty(Z), C^\infty(Z)). \quad (2.16)$$

Since every differential operator is continuous with respect to the Fréchet topology of $C^\infty(Z)$, a complex \mathcal{D}^* is a subcomplex of a complex $B_c^*(C^\infty(Z), C^\infty(Z))$ of continuous Hochschild cochains. Cohomology $H^*(\mathcal{D}^*)$ of the complex \mathcal{D}^* is called the differential Hochschild cohomology. One can show that continuous and differential Hochschild cohomology groups are isomorphic [39, 43, 42]:

$$H_c^r(C^\infty(Z), C^\infty(Z)) = H^r(\mathcal{D}^*).$$

As a consequence, one comes to the following result [39, 42].

THEOREM 2.5: Every continuous deformation of $C^\infty(Z)$ is equivalent to the differential one. A continuous equivalence between two differential deformations of $C^\infty(Z)$ is differential (i.e., the maps ϕ_i (2.4) are linear differential operators). \square

Let us turn to the star-product $*$ (2.13) – (2.14) on a Poisson manifold $(Z, \{\cdot, \cdot\})$ defined by the differential deformation (2.13) of a real ring $C^\infty(Z)$. It is a Lie deformation of the Poisson algebra $C^\infty(Z)$. Of course, an arbitrary deformation (2.13) need not be a star-product. A key point is that, in contradistinction with the Hochschild cohomology, the needed Chevalley–Eilenberg cohomology is small [52].

LEMMA 2.6: Any multivector field $\vartheta \in \mathcal{T}_r(Z)$ (4.17) of degree r defines a differential Hochschild r -cocycle

$$\vartheta(f_1, \dots, f_r) = \vartheta[df_1 \wedge \dots \wedge df_r] \in \mathcal{D}^r, \quad (2.17)$$

where \lfloor is the right interior product (4.25). \square

Thus, there is the inclusion

$$\mathcal{T}_r(Z) \subset \mathcal{D}^*. \quad (2.18)$$

Moreover, any differential Hochschild cocycle is cohomologous to some multi-vector field (2.17) as follows [24, 52].

THEOREM 2.7: The inclusion (2.18) induces $C^\infty(Z)$ -module isomorphisms

$$H^r(\mathcal{D}^*) = \mathcal{T}_r(Z). \quad (2.19)$$

\square

Remark 2.8: By virtue of Theorem 2.7, any Hochschild two-cocycle is cohomologous to its skew-symmetric part. Therefore, we can restrict our consideration to deformations of $C^\infty(Z)$ whose infinitesimal is the Poisson bracket

$$\alpha_1(f, f') = \{f, f'\}, \quad f, f' \in C^\infty(Z), \quad (2.20)$$

in order to obtain a star-product. The fact that a regular Poisson bracket is a Hochschild two-cocycle is easily justified when it is written with respect to local Darboux coordinates (p_i, q^i) and, thus, is a sum of cup-products $\partial^i \smile \partial_i$ of mutually commutative vector fields ∂^i and ∂_i (Example 2.4). \square

Let us consider star-products on a symplectic manifold (Z, Ω) .

The Moyal product on $Z = \mathbb{R}^{2m}$ was the first example of a star-product [23, 25]. Let \mathbb{R}^{2m} be provided with the coordinates (q^i, p_i) and the canonical symplectic form $\Omega = dp_i \wedge dq^i$. Let us consider a differential ring deformation of $C^\infty(\mathbb{R}^{2m})$ whose infinitesimal is the Poisson bracket (2.8). This infinitesimal is a sum of cup-products of mutually commutative vector fields ∂^i and ∂_i . Then, generalizing Example 2.4, one can show that such a deformation exists and it is given by the expression

$$\begin{aligned} f * f' &= \exp\left[\frac{h}{2}\{f, f'\}\right] = f \exp\left[\frac{h}{2}(\overleftarrow{\partial} \overrightarrow{\partial}_i - \overleftarrow{\partial}_i \overrightarrow{\partial})\right] f' = \\ &= \sum_{k=0} \frac{h^k}{k!} \sum_{r=0}^k (-1)^r (\partial_{i_1} \cdots \partial_{i_r} \partial^{i_{r+1}} \cdots \partial^{i_k} f) (\partial^{i_1} \cdots \partial^{i_r} \partial_{i_{r+1}} \cdots \partial_{i_k} f'). \end{aligned} \quad (2.21)$$

This is a star-product, called the Moyal product. Since the de Rham cohomology group $H_{\text{DR}}^2(\mathbb{R}^{2m})$ of \mathbb{R}^{2m} is trivial, all star-products on $(\mathbb{R}^{2m}, \Omega)$ are equivalent to the Moyal one (2.21). This star-product defines the corresponding Lie deformation (2.15).

Fedosov's deformation quantization [12] generalizes a construction of the Moyal product (2.21) to an arbitrary symplectic manifold as follows.

THEOREM 2.8: Any symplectic manifold admits a star-product [9], e.g., Fedosov's one [12]. \square

THEOREM 2.9: Any star-product on a symplectic manifold is equivalent to some Fedosov star-product [3, 55]. \square

THEOREM 2.10: The equivalence classes of star-products on a symplectic manifold constitute an affine space modelled on the linear space $H^2(Z)[[h]]$ of power series in h whose coefficients are elements of the de Rham cohomology group $H_{\text{DR}}^2(Z)$ of Z [8, 22, 40]. \square

Namely, given two different star-products $*$ and $*'$, one associate to them a unique Čech cohomology class

$$t(*', *) \in H^2(Z; \mathbb{R})[[h]], \quad (2.22)$$

called Deligne's relative class. It is defined as follows [8, 22].

LEMMA 2.11: Star-products $*$ and $*'$ are equivalent if they are equivalent as ring deformations, i.e.,

$$\begin{aligned} \phi(f * g) &= \phi f *' \phi g, & f, g &\in C^\infty(Z), \\ \phi &= \text{Id} + \sum_{r=1} h^r \phi_r, \end{aligned} \quad (2.23)$$

where ϕ_r are differential operators. \square

One calls ϕ (2.23) the formal differential operator.

THEOREM 2.12: Let (Z, Ω) be a symplectic manifold, and let us suppose that the Čech cohomology group $H^2(Z; \mathbb{R})$ is trivial. Then any two star-products on Z are equivalent. \square

THEOREM 2.13: Let $*$ be a star-product on (Z, Ω) , and let us suppose that $H^1(Z; \mathbb{R}) = 0$. Then any self-equivalence ϕ (2.23) of $*$ is inner, i.e.,

$$\phi = \exp\{\text{ad}_* \gamma\},$$

for some $\gamma \in C^\infty(Z)[[h]]$, where

$$\text{ad}_* \gamma(f) = [g, f]_* = g * f - f * g, \quad f \in C^\infty(Z)[[h]],$$

is the star adjoint representation. \square

Let $\{U_i\}$ be a locally finite open cover of Z by Darboux coordinate charts such that U_i and all their non-empty intersections are contractible. Let us denote $C_i = C^\infty(U_i)$. A star-product $*$ in $C^\infty(Z)[[h]]$ certainly yields a star-product $*$ in C_i on a symplectic manifold (U_i, Ω) . Let $*$ and $*'$ be two star-products on Z . By virtue of Theorem 2.12, their restrictions to U_i are equivalent star products, i.e., there exists the formal differential operator (2.23):

$$\phi_i : C_i[[h]] \rightarrow C_i[[h]]$$

such that

$$\phi_i(f * g) = \phi_i f * \phi_i g, \quad f, g \in C_i[[h]].$$

On $U_i \cap U_j$, we accordingly have a self-equivalence $\phi_j^{-1} \circ \phi_i$ of $*$ in $C_{ij}[[h]]$. By virtue of Theorem 2.13, this self-equivalence is inner, i.e., there exists an element $\gamma_{ji} = -\gamma_{ij} \in C_{ij}[[h]]$ such that

$$\phi_j^{-1} \circ \phi_i = \exp\{\text{ad}_* \gamma_{ji}\}.$$

On $U_i \cap U_j \cap U_k$, the composition

$$\text{ad}_* \gamma_{kji} = \text{ad}_* \gamma_{ik} \circ \text{ad}_* \gamma_{kj} \circ \text{ad}_* \gamma_{ji}$$

is the identity morphism of $C_{ijk}[[h]]$ and, consequently, is represented by an element γ_{kji} in the center $\mathbb{R}[[h]]$ of $C_{ijk}[[h]]$. The standard arguments show that the set of the elements γ_{kji} define a Čech two-cocycle with values in $\mathbb{R}[[h]]$. Its cohomology class $[\gamma_{kji}] \in H^2(Z; \mathbb{R})$ is desired Deligne's relative class (2.22).

THEOREM 2.14: If $*$, $'$ and $''$ are three star-products on (Z, Ω) , then

$$t(*'', *) = t(*'', *) + t(*', *).$$

□

A moduli space of equivalent star-products on a symplectic manifold (Z, Ω) is usually identified with

$$\frac{1}{h}[\Omega] + H^2(Z)[[h]].$$

Let us point out an isomorphism

$$H^2(C^\infty(Z)[[h]], C^\infty(Z)[[h]]) = Z^2(Z) + H^2(Z)[[h]], \quad (2.24)$$

where $Z^2(Z)$ is a space of closed two-forms on Z .

2.3 Kontsevich's deformation quantization

Let us turn to star-products on Poisson manifolds. A star-product on an arbitrary (smooth, not necessarily regular) Poisson manifold exists. Its general construction was first suggested in [28, 30].

If $(Z, \{, \})$ is a regular Poisson manifold, there exists a tangential star-product on Z [13, 37]. It is defined as follows. Let $\mathcal{S}_{\mathcal{F}}(Z)$ be the center of the Poisson algebra $C^\infty(Z)$. Let us consider a $\mathcal{S}_{\mathcal{F}}(Z)$ -relative deformation of a real ring $C^\infty(Z)$. A star-product on Z and a Lie deformation of $C^\infty(Z)$ which comes from an $\mathcal{S}_{\mathcal{F}}(Z)$ -relative deformation (Definition 2.4) are called tangential because $\mathcal{S}_{\mathcal{F}}(Z)$ consists of functions constant on leaves of the characteristic foliation of $(Z, \{, \})$. A tangential star-product can be introduced either as the tangential version of Vey's work [33, 37] or the straightforward generalization

of Fedosov's deformation quantization of symplectic manifolds to symplectic foliations.

Kontsevich's deformation [28, 30] generalizes the Moyal star-product on \mathbb{R}^{2m} to a generic Poisson structure on \mathbb{R}^{2m} which fails to be regular and does not admit the Darboux coordinates. Recently, Kontsevich's deformation quantization has been extended to an arbitrary Poisson manifold [6, 11, 29]. The key point of Kontsevich's deformation is the formality theorem (Theorem 2.45). This theorem and Theorem 2.16 establish the relations between algebras of multivector fields and multidifferential operators on a smooth manifold.

2.3.1 Differential graded Lie algebras

We start with the relevant algebraic constructions (see, e.g., [7, 10]). Unless otherwise stated, all algebras are over a field \mathbb{K} of characteristic zero. By a graded structure is meant a \mathbb{Z} -graded structure, and the symbol $|\cdot|$ stands for the \mathbb{Z} -graded parity.

A differential graded Lie algebra (henceforth a DGLA) is a differential graded algebra

$$\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}^i,$$

whose multiplication operation is a graded Lie bracket

$$[\cdot, \cdot] : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathcal{G}, \quad [a, b] \in \mathcal{G}^{|a|+|b|},$$

satisfying the relations

$$\begin{aligned} [a, b] &= -(-1)^{|a||b|} [b, a], \quad a, b, c \in \mathcal{G}, \\ (-1)^{|a||c|} [a, [b, c]] &+ (-1)^{|b||a|} [b, [c, a]] + (-1)^{|c||b|} [c, [a, b]] = 0, \end{aligned} \quad (2.25)$$

and the differential d of degree one obeys the graded Leibniz rule

$$d[a, b] = [da, b] + (-1)^{|a|} [a, db]. \quad (2.26)$$

The degree zero part \mathcal{G}^0 and the even part of a DGLA \mathcal{G} are Lie algebras. Any Lie algebra is a DGLA of zero degree with $d = 0$.

A morphism of DGLAs is a graded linear map of degree zero which commutes with differentials and brackets. In particular, it is a cochain morphism.

Let $H^*(\mathcal{G})$ be the cohomology of a DGLA \mathcal{G} (Definition 4.4).

THEOREM 2.15: The cohomology $H^*(\mathcal{G})$ of a DGLA \mathcal{G} is a DGLA with respect to the zero differential $d = 0$ and the bracket

$$[\bar{a}, \bar{b}]_H = \overline{[a, b]}, \quad a, b \in \mathcal{G},$$

where \bar{a} denotes the cohomology class of $a \in \mathcal{G}$. \square

It is evident that the cohomology of a DGLA $H^*(\mathcal{G})$ coincides with $H^*(\mathcal{G})$, i.e., $H^*(H^*(\mathcal{G})) = H^*(\mathcal{G})$.

Every morphism $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$ of DGLAs yields a morphism of the cohomology DGLAs

$$\overline{\Phi} : H^*(\mathcal{G}) \rightarrow H^*(\mathcal{G}'). \quad (2.27)$$

DEFINITION 2.5: A DGLA morphism Φ is called the quasi-isomorphism if the induced morphism $\overline{\Phi}$ (2.27) is an isomorphism. \square

Let us emphasize that the existence of a quasi-isomorphism $\mathcal{G} \rightarrow \mathcal{G}'$ does not imply the existence of the 'quasi-inverse' $\mathcal{G}' \rightarrow \mathcal{G}$ which induces the inverse morphism $\overline{\Phi}^{-1}$.

DEFINITION 2.6: If the quasi-inverse exists, the DGLAs \mathcal{G} and \mathcal{G}' are called quasi-isomorphic. A DGLA \mathcal{G} is called formal if it is quasi-isomorphic to the DGLA $H^*(\mathcal{G})$. \square

A graded coalgebra is a graded vector space \mathfrak{f} provided with the following graded co-operations:

- a graded comultiplication $\Delta : \mathfrak{f} \rightarrow \mathfrak{f} \otimes \mathfrak{f}$ such that

$$\Delta(\mathfrak{f}^i) \subset \bigoplus_{j+k=i} \mathfrak{f}^j \otimes \mathfrak{f}^k;$$

- a counit $\epsilon : \mathfrak{f} \rightarrow \mathbb{K}$ such that $\epsilon(\mathfrak{f}^{i>0}) = 0$.

These operations obey the relations

$$(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \epsilon) \circ \Delta = \text{Id}.$$

A graded coalgebra is called graded cocommutative if $P \circ \Delta = \Delta$, where

$$P(a \otimes b) = (-1)^{|a||b|} b \otimes a, \quad a, b \in \mathfrak{f},$$

is the graded transposition operator.

A coderivation of degree r on a graded coalgebra \mathfrak{f} is a graded linear map $\partial : \mathfrak{f}^i \rightarrow \mathfrak{f}^{i+r}$ which satisfies the co-Leibniz rule:

$$(\Delta \circ \partial)(v) = ((\partial \otimes \text{Id} + (-1)^{r|v|} \text{Id} \otimes \partial) \circ \Delta)(v), \quad v \in \mathfrak{f}.$$

By a codifferential on a graded coalgebra is meant a nilpotent coderivation of degree one.

2.3.2 L_∞ -algebras

The fact that the quasi-inverse of DGLAs need not exists has motivated one to call into play a wider class of algebras, which are L_∞ -algebras.

Let V be a graded vector space and

$$\otimes V = \mathbb{K} \oplus V \oplus \dots \oplus V^{\otimes k} \oplus \dots$$

its tensor algebra. It is a graded vector space such that

$$|v \otimes v'| = |v| + |v'|, \quad v, v' \in V,$$

and $|\lambda| = 0$, $\lambda \in \mathbb{K}$. We call $\otimes_+ V = \otimes V \setminus \mathbb{K}$ the reduced tensor algebra.

A tensor algebra $\otimes V$ is brought into a graded coalgebra with respect to the counit $\epsilon : \otimes V \rightarrow \mathbb{K}$ and the comultiplication

$$\begin{aligned} \Delta(v_1 \otimes \cdots \otimes v_r) &= 1 \otimes (v_1 \otimes \cdots \otimes v_r) + \\ &\sum_{j=1}^{r-1} ((v_1 \otimes \cdots \otimes v_j) \otimes ((v_{j+1} \otimes \cdots \otimes v_r) + (v_1 \otimes \cdots \otimes v_r) \otimes 1. \end{aligned}$$

A graded symmetric algebra $\vee V$ and a graded exterior algebra $\wedge V$ are defined as the the quotients of $\otimes V$ by the two-sided ideals generated by homogeneous elements of the form $v \otimes v' - P(v \otimes v')$ and $v \otimes v' + P(v \otimes v')$, respectively. Accordingly, we have the reduced algebras $\vee_+ V$ and $\wedge_+ V$. A graded symmetric algebra $\vee V$ is brought into a graded coalgebra provided with the comultiplication given on elements of V by

$$\Delta(v) = 1 \otimes v + v \otimes 1,$$

and extended as an algebra homomorphism with respect to the tensor product.

Given a graded vector space V , one can obtain a new graded vector space $V[k]$ by shifting the degree by k , i.e.,

$$V[k]^i = V^{i+k}, \quad |v|[k] = |v| - k, \quad v \in V.$$

Then we have the décalage isomorphism between the graded symmetric and exterior algebras. It is given on the k -symmetric power of V shifted by one by the expression

$$\begin{aligned} \zeta_k : \vee^k V[1] &\rightarrow \wedge^k V[k], \\ \zeta_k : v_1 \vee \cdots \vee v_k &\rightarrow (-1)^{\sum_{i=1}^k (k-i)(|v_i|-1)} v_1 \wedge \cdots \wedge v_k. \end{aligned} \tag{2.28}$$

An L_∞ -algebra is a graded vector space \mathcal{G} endowed with a codifferential D on the reduced symmetric space $\vee_+ \mathcal{G}[1]$.

Let us abbreviate the symbol D_j^i with the projection of D to the component $\vee_+^i \mathcal{G}[1]$ of the target vector space and $\vee_+^j \mathcal{G}[1]$ of the domain space. A coderivation D is proved to be uniquely determined by its components D_r^1 , and it is a codifferential iff

$$\sum_{i=1}^n D_i^1 \circ D_n^i = 0, \quad n \geq 1. \tag{2.29}$$

In particular, $D_1^1 \circ D_1^1 = 0$, i.e., a codifferential D defines a complex on \mathcal{G} .

One can show that any DGLA (\mathcal{G}, d) can be brought into an L_∞ -algebra where

$$D_1^1 a = (-1)^{|a|} da, \quad D_2^1(a \vee b) = (-1)^{|a|(|b|-1)}[a, b], \quad D_n^1 = 0, \quad n \geq 3.$$

An L_∞ -algebra morphism $\Phi : (\mathcal{G}, D) \rightarrow (\mathcal{G}', D')$ is a morphism of graded coalgebras

$$\Phi : \vee_+ \mathcal{G}[1] \rightarrow \vee_+ \mathcal{G}'[1], \tag{2.30}$$

which intertwines the codifferentials, i.e., $\Phi \circ D = D' \circ \Phi$. An L_∞ -algebra morphism is uniquely determined by its components

$$\Phi_j^1 : \check{\vee}_+^j \mathcal{G}[1] \rightarrow \mathcal{G}'[1] \quad (2.31)$$

which obey the relations

$$\sum_{i=1}^n \Phi_i^1 \circ D_n^i = \sum_{i=1}^n D_i'^1 \circ \Phi_n^i, \quad n \geq 1. \quad (2.32)$$

In particular, a morphism Φ of DGLAs induces their morphism $\tilde{\Phi}$ as L_∞ -algebras such that $\tilde{\Phi}_1^1 = \Phi$.

Due to the décalage isomorphism (2.28), the codifferential D defines a sequence $\{l_k\}$ of morphisms

$$l_k : \wedge^k \mathcal{G} \rightarrow \mathcal{G}[2-k], \quad k \geq 1.$$

The relation (2.29) puts an infinite family of conditions on the l_k 's. These conditions imply that $l_1 : \mathcal{G} \rightarrow \mathcal{G}[1]$ is a differential of degree one obeying the graded Leibniz rule. It follows that (\mathcal{G}, l_1) is a complex. The morphism l_2 defines a graded bracket $[\cdot, \cdot] : \wedge^2 \mathcal{G} \rightarrow \mathcal{G}$ on \mathcal{G} compatible with l_1 (i.e., the relation (2.26) holds). This bracket obeys the graded Jacobi identity up to homotopy given by l_3 . Hence, any L_∞ -algebra (\mathcal{G}, D) such that $l_k = 0$ for $k \geq 3$ is a DGLA.

Every L_∞ -algebra morphism Φ provides a cochain morphism Φ_1 between the complexes (\mathcal{G}, l_1) and (\mathcal{G}', l_1') . One says that an L_∞ -morphism Φ (2.30) of L_∞ -algebras is a quasi-isomorphism if Φ_1 is a quasi-isomorphism of complexes (\mathcal{G}, l_1) and (\mathcal{G}', l_1') , i.e., it yields an isomorphism of their cohomology. In contrast with morphisms of DGLAs, any quasi-isomorphism of L_∞ -algebras possesses the quasi-inverse. Thus, quasi-isomorphisms of L_∞ -algebras define an equivalence relation, i.e., two L_∞ -algebras are quasi-isomorphic iff there is an L_∞ -quasi-isomorphism between them. The notion of formality of L_∞ -algebras is formulated similarly to that of DGLAs.

2.3.3 Formality theorem

Let us show that multivector fields on a smooth manifold Z constitute a DGLA. The graded commutative algebra $\mathcal{T}_*(Z)$ of multivector fields is customarily provided with the Schouten–Nijenhuis bracket $[\cdot, \cdot]_{\text{SN}}$ (4.18) which obey the relations (4.19) – (4.21). However, there is another sign convention used in the definition of the Schouten–Nijenhuis bracket [36]. This bracket, denoted by $[\cdot, \cdot]_{\text{SN}'}$, is

$$[\vartheta, v]_{\text{SN}'} = -(-1)^{|\vartheta|} [\vartheta, v]_{\text{SN}}. \quad (2.33)$$

The relation (4.19) for this bracket reads

$$[\vartheta, v]_{\text{SN}'} = -(-1)^{(|\vartheta|-1)(|v|-1)} [v, \vartheta]_{\text{SN}'}. \quad (2.34)$$

The relation (4.20) keeps its form, while the relation (4.21) is replaced with the following one

$$\begin{aligned} & (-1)^{(|\nu|-1)(|\nu|-1)}[\nu, [\vartheta, \nu]_{\text{SN}'}]_{\text{SN}'} + (-1)^{(|\vartheta|-1)(|\nu|-1)}[\vartheta, [\nu, \nu]_{\text{SN}'}]_{\text{SN}'} \\ & + (-1)^{(|\nu|-1)(|\vartheta|-1)}[\nu, [\nu, \vartheta]_{\text{SN}'}]_{\text{SN}'} = 0. \end{aligned} \quad (2.35)$$

The equalities (2.34) and (2.35) show that, with the modified Schouten–Nijenhuis bracket (2.33), the graded vector space

$$\mathcal{V}^* = \mathcal{T}_*(Z)[1] \quad (2.36)$$

of multivector fields on a manifold Z is precisely a graded Lie algebra. This graded Lie algebra is brought into a DGLA by setting the differential d to be identically zero. Clearly, this DGLA is formal.

In particular, let (Z, w) be a Poisson manifold. Then the Poisson bivector w obeys the Maurer–Cartan equation

$$dw + \frac{1}{2}[w, w]_{\text{SN}'} = 0 \quad (2.37)$$

on the DGLA \mathcal{V}^* (2.36).

Let us describe the DGLA of multidifferential operators on a ring $C^\infty(Z)$ of smooth real functions on a manifold Z .

In a general setting, let \mathcal{A} be a \mathbb{K} -ring and $B^*(\mathcal{A}, \mathcal{A})$ its Hochschild complex (4.60). Let us consider the complex

$$\mathcal{C}\ell^* = B^*(\mathcal{A}, \mathcal{A})[1]. \quad (2.38)$$

It inherits the Hochschild coboundary operator (4.61):

$$\begin{aligned} (\delta\phi^k)(a_0, \dots, a_{k+1}) &= a_0\phi^k(a_1, \dots, a_{k+1}) + \\ & \sum_j (-1)^{j+1}\phi^k(a_0, \dots, a_j a_{j+1}, \dots, a_{k+1}) + (-1)^{k+2}\phi^k(a_0, \dots, a_k) a_{k+1}, \end{aligned} \quad (2.39)$$

the composition product (4.67):

$$\begin{aligned} \phi^m \circ \phi^n(a_0, \dots, a_{m+n}) &= \\ & \sum_{i=0}^m (-1)^{in}\phi^m(a_0, \dots, a_{i-1}, \phi^n(a_i, \dots, a_{n+i}), a_{n+i+1}, \dots, a_{m+n}), \end{aligned} \quad (2.40)$$

and the Gerstenhaber bracket (4.68):

$$[\phi, \phi']_{\text{G}} = \phi \circ \phi' - (-1)^{|\phi||\phi'|}\phi' \circ \phi, \quad \phi, \phi' \in \mathcal{C}\ell^*. \quad (2.41)$$

One can show that this bracket obeys the graded Jacobi identity (2.25). Thus, $(\mathcal{C}\ell^*, [\cdot, \cdot]_{\text{G}})$ is a graded Lie algebra. Furthermore, let a one-cocycle

$$\phi^1(a_0, a_1) = \mathfrak{m}(a_0, a_1) = a_0 a_1$$

be the multiplication in \mathcal{A} . We consider the operator

$$d_{\mathfrak{m}}\phi = [\mathfrak{m}, \phi]_{\mathcal{G}}, \quad \phi \in \mathcal{C}\ell^*, \quad (2.42)$$

of degree one on $\mathcal{C}\ell^*$. A direct computation shows that

$$d_{\mathfrak{m}}\phi = (-1)^{|\phi|}\delta\phi, \quad \phi \in \mathcal{C}\ell^*,$$

and the relation (2.26) holds. Then the complex $\mathcal{C}\ell^*$ (2.38) is a DGLA with respect to the bracket (2.41) and the differential (2.42).

Let $\mathcal{A} = C^\infty(Z)$, and let us consider a subcomplex \mathcal{D}^* (2.16) of the complex $B^*(C^\infty(Z), C^\infty(Z))[1]$ whose cochains are multidifferential operators on $C^\infty(Z)$. This subcomplex is closed with respect both to the Gerstenhaber bracket (2.41) and the action of $d_{\mathfrak{m}}$. Thus, it is a desired DGLA of multidifferential operators.

Given a bidifferential operator $\alpha \in \mathcal{D}^1$, one can think of

$$f * f' = (\mathfrak{m} + \alpha)(f, f') = ff' + \alpha(f, f'), \quad f, f' \in C^\infty(Z), \quad (2.43)$$

as being a deformation of the original product \mathfrak{m} in $C^\infty(Z)$. One can show that the associativity constraint on this deformation is given by the equality

$$[\mathfrak{m} + \alpha, \mathfrak{m} + \alpha]_{\mathcal{G}} = 0,$$

which takes a form of the Maurer–Cartan equation

$$d_{\mathfrak{m}}\alpha + \frac{1}{2}[\alpha, \alpha]_{\mathcal{G}} = 0. \quad (2.44)$$

Now, the goal is to construct a morphism of the DGLA \mathcal{V}^* (2.36) of multivector fields to the DGLA \mathcal{D}^* of multidifferential operators which intertwines their differential graded Lie algebra structures and solutions of the Maurer–Cartan equations (2.37) and (2.44).

One has proved the following [24].

THEOREM 2.16: For any smooth manifold Z , there is an isomorphism between the cohomology $H^*(\mathcal{D}^*)$ of the algebra \mathcal{D} and the algebra \mathcal{V}^* . Since \mathcal{V}^* coincides with its cohomology $H^*(\mathcal{V}^*)$, we have an isomorphism

$$H^*(\mathcal{D}^*) \rightarrow \mathcal{V}^* \cong H^*(\mathcal{V}^*). \quad (2.45)$$

□

The next step is the above mentioned formality theorem.

THEOREM 2.17: The DGLA \mathcal{D}^* of multidifferential operators on a smooth manifold Z is formal. □

It follows that there exists a quasi-isomorphism of the DGLA \mathcal{D}^* to $H^*(\mathcal{D}^*) \cong \mathcal{V}^*$ and, consequently, there is the inverse L_∞ -quasi-isomorphism \mathcal{U} of \mathcal{V}^* to \mathcal{D}^* .

Remark 2.9: There is a natural quasi-isomorphism $\mathcal{U}_1^{(0)}$ of complex \mathcal{V}^* to the complex \mathcal{D}^* . This morphism associates to a multivector field $\vartheta_0 \wedge \cdots \wedge \vartheta_n$

the multidifferential operator whose action on functions $f_0, \dots, f_n \in C^\infty(Z)$ is given by the expression

$$\frac{1}{(n+1)!} \sum_s \text{sgn}(s) \vartheta_{s(0)}(f_0) \cdots \vartheta_{s(n)}(f_n)$$

where s runs through all permutations of the numbers $(0, \dots, n)$ and $\text{sgn}(s)$ is the sign of a permutation s . For instance, $\mathcal{U}_1^{(0)}$ assigns to a Poisson bivector w the Poisson bracket $\frac{1}{2}\{\cdot, \cdot\}$. However, the morphism $\mathcal{U}_1^{(0)}$ fails to preserve the Lie structure. \square

An L_∞ -quasi-isomorphism \mathcal{U} of \mathcal{V}^* to \mathcal{D}^* intertwines their differential graded Lie algebra structures and solutions of the Maurer–Cartan equations (2.37) and (2.44). Let us note that this L_∞ -quasi-isomorphism fails to be canonical. It is represented by a power series whose first term of differential operators of minimal order coincides with the morphism $\mathcal{U}_1^{(0)}$ in Remark 2.9. This morphism associates to each Poisson bivector field $w \in \mathcal{V}^1$ on Z a certain bidifferential operator $\alpha_w \in \mathcal{D}^1$ which obeys the Maurer–Cartan equation (2.44) and, thus, defines a differential associative deformation $\mathfrak{m} + \alpha_w$ of the ring $C^\infty(Z)$.

An L_∞ -quasi-isomorphism \mathcal{U}_1 of \mathcal{V}^1 to \mathcal{D}^1 on $Z = \mathbb{R}^r$ in the explicit form has been obtained in [28, 30].

3 Deformation quantization on jet manifolds

Let X be a smooth manifold of dimension $n > 1$ and $C^\infty(X)$ a real ring of real smooth functions on X . As was mentioned above, we restrict our study of deformation quantization on X to differential deformations whose terms of non-zero degree in a deformation parameters are multidifferential operators.

As was mentioned above, jet formalism provides the adequate formulation of theory of differential operators and differential equations on manifolds and fibre bundles [5, 19, 31]. Therefore, we develop an idea in [6, 7, 20] and aim to describe deformation quantization in terms of jets.

Since deformation terms are multilinear differential operators, but not the linear ones, you follow the technique of jets of sections of fibre bundles over X [19, 47, 48]. Note that jets of sections are the particular jets of maps [27, 44]. If these are sections of vector bundles, their jets coincide with jets of $C^\infty(X)$ -modules [31, 35] which are representative objects of linear differential operators on modules [31, 46].

Following Definition 3.2 of differential operators, we formulate multidifferential operators in the jet terms (Definition 3.3) and define their infinite order jet prolongation on the infinite order jet manifold $J^\infty F$ (3.14) of jets of smooth real functions on a manifold X .

3.1 Multidifferential operators on $C^\infty(X)$

In the framework of formalism of jets of sections of fibre bundles (Appendix), a real vector space $C^\infty(X)$ of smooth real functions on a manifold X is represented as a structure module $F(X)$ of global sections of a trivial vector bundle

$$F = X \times \mathbb{R} \rightarrow X, \quad (3.1)$$

provided with bundle coordinates (x^μ, t) where a coordinate t possesses the identity transition functions, i.e., it is a global coordinate on \mathbb{R} . Without the loss of generality, we assume that it is the Cartesian coordinate on the linear space of real number \mathbb{R} .

The tangent bundle, vertical tangent bundle and the cotangent bundle of the fibre bundle F (3.1) are respectively the following:

$$TF = TX \times \mathbb{R} \times \mathbb{R} = TX \times_X F \times_X F, \quad (3.2)$$

$$VF = F \times \mathbb{R} = F \times F, \quad (3.3)$$

$$T^*F = T^*X \times \mathbb{R} \times \mathbb{R} = T^*X \times_X F \times_X F. \quad (3.4)$$

Let J^1F be the first order jet manifold of a sections of the fibre bundle F (3.1). It is provided with the adapted coordinates $(x^\lambda, t, t_\lambda)$ (4.28) possessing transition functions

$$t'_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} t_\mu. \quad (3.5)$$

Comparing the expressions (3.5) and (4.2) shows that the first order jet manifold J^1F is isomorphic to a bundle product

$$J^1F = F \times_X T^*X. \quad (3.6)$$

Herewith, the jet bundles (4.29):

$$\pi^1 : J^1F \rightarrow X, \quad \pi_0^1 : J^1F \rightarrow F, \quad (3.7)$$

are vector bundles.

Accordingly, the canonical imbeddings (4.30) – (4.31) take a form

$$\lambda_1 : J^1F \xrightarrow{F} T^*X \otimes_F TF, \quad \lambda_1 = dx^\lambda \otimes (\partial_\lambda + t_\lambda \partial_t) = dx^\lambda \otimes d_\lambda, \quad (3.8)$$

$$\theta_1 : J^1F \rightarrow T^*F \otimes_F VF, \quad \theta_1 = (dt - t_\lambda dx^\lambda) \otimes \partial_t = \theta \otimes \partial_t, \quad (3.9)$$

where d_λ are total derivatives and θ is the contact forms. Identifying the jet manifold J^1F to its images under the canonical morphisms (3.8) and (3.9), one can represent jets $j_x^1 s = (x^\lambda, t, t_\mu)$ by tangent-valued forms

$$dx^\lambda \otimes (\partial_t + t_\lambda \partial_y), \quad (dt - t_\lambda dx^\lambda) \otimes \partial_t. \quad (3.10)$$

Let $J^r F$ now be the r -order jet manifold of the fibre bundle F (3.1). It is provided with the adapted coordinates (x^λ, t_Λ) (4.35) possessing transition functions (4.36):

$$t'_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu t'_\Lambda. \quad (3.11)$$

where the total derivative d_λ read

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} t_{\Lambda+\lambda} \frac{\partial}{\partial t_\Lambda}. \quad (3.12)$$

Finite order jet manifolds $J^r F$ constitute the inverse system (4.47):

$$X \xleftarrow{\pi} F \xleftarrow{\pi_0^1} \dots \xleftarrow{\pi_{r-1}^r} J^{r-1} F \xleftarrow{\pi_{r-1}^r} J^r F \xleftarrow{\pi_r^r} \dots, \quad (3.13)$$

whose projective limit (4.48) is the infinite order jet manifold

$$J^\infty F = \varprojlim J^r F, \quad (3.14)$$

together with surjections π^∞ , π_0^∞ and π_k^∞ (4.49). It is a paracompact Fréchet manifold. A manifold atlas $\{U, x^\lambda\}$ of X yields the manifold coordinate atlas

$$\{(\pi^\infty)^{-1}(U), (x^\lambda, t_\Lambda)\}, \quad 0 \leq |\Lambda|,$$

of $J^\infty F$, together with the transition functions (3.11) where d_λ is the total derivative (3.12).

Remark 3.1: One can think of $J^\infty F$ as being a space of infinite order jets of real smooth functions on a manifold X . \square

The inverse system (3.13) yields the direct system (4.43) of algebras of exterior form on finite order jet manifolds and, in particular, the direct system (4.54):

$$C^\infty(X) \xrightarrow{\pi^*} C^\infty(F) \xrightarrow{\pi_0^{1*}} C^\infty(J^1 F) \longrightarrow \dots \longrightarrow C^\infty(J^r F) \longrightarrow \dots \quad (3.15)$$

of vector spaces $C^\infty(J^r F) = \mathcal{O}_r^0$ of smooth real functions on finite order jet manifolds. Let

$$C^\infty(J^\infty F) = \mathcal{O}_\infty^0 \quad (3.16)$$

be its direct limit. It consists of smooth real functions on finite order jet manifolds modulo the pull-back identification.

DEFINITION 3.1: We agree to call elements of the direct limit (3.16) the smooth real functions of finite jet order on the infinite order jet space $J^\infty F$. \square

Turn now to the notion of a k -order $C^\infty(X)$ -valued differential operator on a real vector space $C^\infty(X)$. In accordance with Definition 4.1 it defines as a section of the pull-back bundle (4.44):

$$J^k F \times_X F = J^k F \times \mathbb{R} \rightarrow J^k F, \quad (3.17)$$

(cf. (3.1)) which is smooth real function on the jet manifold $J^k F$. Thus, we come to the following.

DEFINITION 3.2: The k -order $C^\infty(X)$ -valued differential operators on a real vector space $C^\infty(X)$ are given by smooth real functions \mathcal{E} on the jet manifold $J^k F$. \square

Furthermore, by very definition of the direct limit (3.16) of the direct system (3.15), there is a pull-back monomorphism

$$\pi_k^{\infty*} : C^\infty(J^k F) \rightarrow C^\infty(J^\infty F). \quad (3.18)$$

Consequently, we come to the following.

THEOREM 3.1: Any finite order $C^\infty(X)$ -valued differential operator on $C^\infty(X)$ (Definition 3.2) is represented by the pull-back smooth real function

$$\pi_k^{\infty*} \mathcal{E} \in C^\infty(J^\infty F) \quad (3.19)$$

of finite jet order on the infinite order jet manifold $J^\infty Y$. \square

Let us generalize this construction to multidifferential operators.

DEFINITION 3.3: A $C^\infty(X)$ -valued m -multidifferential operator on $C^\infty(X)$ is defined to be a m -linear smooth real function of finite jet order on some product of the infinite order jet space

$$\Pi^m = \times_X^m J^\infty F, \quad (3.20)$$

which is the pull-back of a m -linear smooth real function on a product of finite order jet spaces

$$\times_X^m J^* F = J^{k_1} F \times_X \cdots \times_X J^{k_m} F. \quad (3.21)$$

\square

Example 3.2: Any multivector field ϑ (4.17) on a manifold X yields a multidifferential operator on $C^\infty(X)$ defined by a function

$$\vartheta = \frac{1}{m} \vartheta^{\lambda_1 \cdots \lambda_m} t_{\lambda_1}^1 \cdots t_{\lambda_m}^m \quad (3.22)$$

on the product Π^m (3.20). In these terms, the Schouten–Nijenhuis bracket (4.18) reads

$$\begin{aligned} [\vartheta, v]_{\text{SN}} &= \vartheta \bullet v + (-1)^{ms} v \bullet \vartheta, \\ \vartheta \bullet v &= \frac{m}{m!s!} (\vartheta^{\mu\lambda_2 \cdots \lambda_m} d_\mu v^{\alpha_1 \cdots \alpha_s}, t_{\lambda_2}^1, \dots, t_{\lambda_m}^{m-1}, t_{\alpha_1}^m, \dots, t_{\alpha_s}^{m+s-1}). \end{aligned} \quad (3.23)$$

\square

Example 3.3: Let (X, w) be a Poisson manifold. Then the Poisson bracket $\{, \}$ (2.12) is a bidifferential operator on $C^\infty(X)$ given by a function

$$w = \frac{1}{2} w^{\mu\nu} t_\mu^1 t_\nu^2, \quad w^{\mu\lambda_1} \partial_\mu w^{\lambda_2\lambda_3} \partial_{\lambda_1} \wedge \partial_{\lambda_2} \wedge \partial_{\lambda_3} = 0, \quad (3.24)$$

on Π^2 . \square

Considering $C^\infty(X)$ -valued multidifferential operators, one can say something more. There are both a canonical bundle isomorphism

$$J^k(\times_X^m F) = \times_X^m J^k F \quad (3.25)$$

and a fibration

$$\times_X^m J^{\max(k_i)} F \rightarrow J^{k_1} F \times_X \cdots \times_X J^{k_m} F. \quad (3.26)$$

Any smooth real function on the product (3.21) yields the pull-back function on the product (3.26) and, consequently, on the jet manifold (3.25). This function gives rise again to the pull-back function on

$$\Pi^m = J^\infty(\times_X^m F) \cong \times_X^m J^\infty F. \quad (3.27)$$

Thus, we come to the following.

THEOREM 3.2: Any $C^\infty(X)$ -valued m -multilinear k -order differential operator on $C^\infty(X)$ is defined by a m -linear smooth real function on the k -order jet manifold (3.26), and it is represented by the pull-back function

$$\phi^m(x, t^1, \dots, t^m) = \phi^{\Lambda_1 \dots \Lambda_m}(x) t_{\Lambda_1}^1 \cdots t_{\Lambda_m}^m, \quad |\Lambda_i| \leq k,$$

of finite jet order on the infinite order jet manifold (3.27). \square

Let $D^m \subset C^\infty(\Pi^m)$ denote a set of these functions. In accordance with Theorem 3.2, there is one-to-one correspondence between D^m and the set \mathcal{D}^m (2.16) of multidifferential operators on $C^\infty(X)$. By analogy with the subcomplex \mathcal{D}^* of the Hochschild complex $B^*(C^\infty(X), C^\infty(X))$ (4.60), let us define the Hochschild complex

$$D^1 \xrightarrow{\delta^1} D^2 \xrightarrow{\delta^2} \cdots \longrightarrow D^m \xrightarrow{\delta^m} \cdots, \quad (3.28)$$

$$\begin{aligned} (\delta^m \phi^m)(x, t^1, \dots, t^{m+1}) &= t_1 \phi^m(t_2, \dots, t^{m+1}) + \\ &\sum_j (-1)^j \phi^m(t^1, \dots, t^j a t^{j+1}, \dots, t^{m+1}) + (-1)^{m+1} \phi^m(t^1, \dots, t^m) t^{m+1}. \end{aligned} \quad (3.29)$$

For instance, we have

$$\delta^1 f^1(t, t') = t f^1(t') - f^1(t t') + f^1(t) t', \quad (3.30)$$

$$\begin{aligned} \delta^2 f^2(t, t', t'') &= t f^2(t', t'') - f^2(t t', t'') + \\ &f^2(t, t' t'') - f^2(t, t') t''. \end{aligned} \quad (3.31)$$

The complex D^* (3.28) is provided with the composition product (4.67):

$$\begin{aligned} \phi^m \circ \phi^n(t^1, \dots, t^{m+n-1}) = \\ \sum_{i=1}^m (-1)^{(i-1)(n-1)} \phi^m(t^1, \dots, t^{i-1}, \phi^n(t^i, \dots, t^{n+i-1}), t^{n+i}, \dots, t^{m+n-1}), \end{aligned} \quad (3.32)$$

and the Gerstenhaber bracket (4.68):

$$[\phi, \phi']_G = \phi \circ \phi' - (-1)^{|\phi||\phi'|} \phi' \circ \phi. \quad (3.33)$$

One can show that the bracket (3.33) obeys the graded Jacobi identity (2.25). Thus, $(D^*, [\cdot, \cdot]_G)$ is a graded Lie algebra. Furthermore, let us consider a one-cocycle

$$\phi^1(t, t') = \mathbf{m}(t, t') = tt', \quad (3.34)$$

and introduce the operator

$$d_{\mathbf{m}}\phi = [\mathbf{m}, \phi]_G \quad (3.35)$$

of degree one on D^* . A direct computation shows that

$$d_{\mathbf{m}}\phi = (-1)^{(|\phi|-1)} \delta^{|\phi|} \phi,$$

and the relation (2.26) holds. The bracket (3.33) and the differential (3.35) bring the complex D^* (3.28) into a DGLA, isomorphic to the DGLA $\mathcal{C}\ell^*$ (2.38). Let $H^*(D^*)$ denote its cohomology.

As was mentioned above, any multivector field $\vartheta \in \mathcal{T}_*(X)$ (4.17) on a manifold X yields the multidifferential operator ϑ (3.22) (Example 3.2). Let \mathcal{T}^* denote its subset in D^* (Remark 2.9). It is readily observed that

$$d_{\mathbf{m}}\vartheta = 0 \quad (3.36)$$

for any multivector field ϑ (3.22). It follows that \mathcal{T}^* is a subcomplex of the Hochschild complex D^* (3.28) whose coboundary operators equal zero. It follows that there is a natural bijection between cohomology $H^*(\mathcal{T}^*)$ of this complex and \mathcal{T}^* itself. The modification $[\cdot, \cdot]_{\text{SN}}$ (2.33) of the Schouten–Nijenhuis bracket (3.23) and the coboundary operator $d_{\mathbf{m}}$ (3.36) bring the complex \mathcal{T}^* into a DGLA, isomorphic to the DGLA \mathcal{V}^* (2.36). The cohomology $H^*(\mathcal{T}^*)$ of a DGLA \mathcal{T}^* also is a DGLA (Theorem 2.15). As we have shown above, there is an isomorphism of DGLAs $\mathcal{T}^* = H^*(\mathcal{T}^*)$. Consequently, a DLA \mathcal{T}^* is formal (Definition 2.6).

Herewith, it should be emphasized that a monomorphism of cochain complexes $\mathcal{T}^* \rightarrow D^*$ (cf. (2.18)) fails to be a morphisms of DGLAs because the Schouten–Nijenhuis bracket (3.23) in \mathcal{T}^* is not induced by the Gerstenhaber bracket (3.33) in D^* (cf. Remark 2.9). At the same time, Theorem 2.16 states that there exists a quasi-isomorphism (Definition 2.5)

$$D^* \rightarrow \mathcal{T}^* = H^*(\mathcal{T}^*), \quad (3.37)$$

that is, we have an isomorphism

$$H^*(D^*) = H^*(\mathcal{T}^*) = \mathcal{T}^* \quad (3.38)$$

(cf. (2.19)). In accordance with Theorem 2.17, a DGLA D^* is formal, i.e., there exist both the quasi-isomorphism (3.37) of a DGLA \mathcal{D}^* to $H^*(D^*) = \mathcal{T}^*$ and the inverse L_∞ -quasi-isomorphism $\mathcal{T}^* \rightarrow D^*$.

3.2 Deformations of $C^\infty(X)$

Turn now to deformations of a real ring $C^\infty(X)$. In order to describe them in terms of jets of functions, let us treat a multiplication

$$(f, f') \rightarrow ff', \quad f, f' \in C^\infty(X),$$

in $C^\infty(X)$ as a bidifferential operator of zero order. In accordance with Definition 3.3, it is represented by a real bilinear function (3.34):

$$\mathcal{E}_0(x^\mu, t, t_\Lambda, t', t'_\Lambda) = m(t, t') = tt', \quad (3.39)$$

on the product Π^2 (3.20). It is the pull-back of a real smooth bilinear function

$$\mathcal{E}_0(x^\mu, t, t') = tt' \quad (3.40)$$

on the product $F \times_X F$ (3.21).

Given a deformation parameter h , let $\mathbb{R}[[h]]$ be a real ring of power series in h . It is defined as follows. Let $\mathbb{R}[h]$ be a real commutative ring of polynomials in a quantity h . Let $h^k \mathbb{R}[h]$, $k = 1, 2, \dots$, denote its two-sided ideal generated by an element $h^k \in \mathbb{R}[h]$. Then the quotient

$$\mathbb{R}[h]^{k-1} = \mathbb{R}[h]/h^k \mathbb{R}[h] \quad (3.41)$$

is a real commutative ring, whose ring space is a smooth manifold \mathbb{R}^k . In particular, $\mathbb{R}[h]^0 = \mathbb{R}$ (Remark 2.2).

The ring (3.41) also is the quotient

$$\mathbb{R}[h]^{k-1} = \mathbb{R}[h]^{k+r}/h^k \mathbb{R}[h]^{k+r},$$

and we have a ring epimorphism

$$\xi_k^{k+1} : \mathbb{R}[h]^{k+1} \rightarrow \mathbb{R}[h]^k, \quad k \in \mathbb{N}. \quad (3.42)$$

This epimorphism also is a fibration of ring manifolds $\mathbb{R}^{k+2} \rightarrow \mathbb{R}^{k+1}$.

The rings $\mathbb{R}[h]^k$ (3.41) together with epimorphisms ξ_k^{k+1} constitute the inverse system of real rings

$$\mathbb{R} \longleftarrow \mathbb{R}[h]^1 \longleftarrow \dots \longleftarrow \mathbb{R}[h]^k \longleftarrow \dots. \quad (3.43)$$

Its inverse image is the above mentioned real ring $\mathbb{R}[[h]]$ of power series in h together with epimorphisms

$$\xi_k^\infty : \mathbb{R}[[h]] \rightarrow \mathbb{R}[h]^k, \quad \xi_r^\infty = \xi_r^k \circ \xi_k^\infty. \quad (3.44)$$

In particular,

$$\xi_0^\infty : \mathbb{R}[[h]] \rightarrow \mathbb{R}[h]^0. \quad (3.45)$$

Since the inverse system (3.43) also is an inverse system of smooth fibre bundles over a point, its inverse image $\mathbb{R}[[h]]$ is provided with the inverse limit topology. This is the coarsest topology such that the surjections ξ_k^∞ (3.44) are continuous. The base of open sets of this topology in $\mathbb{R}[[h]]$ consists of the inverse images of \mathbb{R}^{k+1} , $k \in \mathbb{N}$, under the mappings (3.44). This topology makes $\mathbb{R}[[h]]$ into a paracompact Fréchet manifold \mathbb{R}^∞ possessing global manifold coordinates $(t^k, k \in \mathbb{N})$.

A $C^\infty(X)$ -ring $C^\infty(X)[[h]]$ of power series in h is defined as a tensor product over \mathbb{R} of real rings $C^\infty(X)$ and $\mathbb{R}[[h]]$:

$$C^\infty(X)[[h]] = C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}[[h]]. \quad (3.46)$$

It is endowed with product

$$(f \otimes s)(f' \otimes s') = (ff') \otimes (ss'),$$

together with the canonical isomorphisms

$$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}[[h]] = \mathbb{R}[[h]], \quad (3.47)$$

$$C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R} = C^\infty(X). \quad (3.48)$$

The isomorphism (3.47) provides $C^\infty(X)[[h]]$ with the structure of $\mathbb{R}[[h]]$ -ring, whereas the isomorphism (3.48) yields the canonical monomorphism

$$C^\infty(X) \rightarrow C^\infty(X) \otimes_{\mathbb{R}} \mathbf{1} = C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R} \subset C^\infty(X)[[h]]. \quad (3.49)$$

We have also an epimorphism of real rings

$$\xi_0 = \text{Id} \otimes \xi_0^\infty : C^\infty(X)[[h]] \rightarrow C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R} = C^\infty(X), \quad (3.50)$$

where ξ_0^∞ is the epimorphism (3.45).

In order to define multidifferential operators on $C^\infty(X)[[h]]$ let us start with $\mathbb{R}[[h]]$ -linear morphisms

$$\phi : C^\infty(X)[[h]] \rightarrow C^\infty(X)[[h]] \quad (3.51)$$

of an $\mathbb{R}[[h]]$ -module $C^\infty(X)[[h]]$. Obviously, such a morphism ϕ is defined in full by its restriction to the subring $C^\infty(X)$ (3.49) of $C^\infty(X)[[h]]$. This restriction reads

$$\phi(f) = \sum_{r \in \mathbb{N}} h^r \phi_r(f), \quad f \in C^\infty(X), \quad (3.52)$$

where ϕ_k are \mathbb{R} -linear morphisms of a real vector space $C^\infty(X)$. The morphism ϕ (3.51) is said to be an automorphism of $C^\infty(X)[[h]]$ if it is its $\mathbb{R}[[h]]$ -isomorphism.

If ϕ (3.51) is an automorphism of $C^\infty(X)[[h]]$, then $\phi_0 = \xi_0 \circ \phi$ is an automorphism of a real vector space $C^\infty(X)$. Such an automorphism is a multiplication

$$\phi_0(f) = gf, \quad f \in C^\infty(X), \quad (3.53)$$

where g is an invertible element of $C^\infty(X)$. Then any automorphism ϕ (3.51) of $C^\infty(X)[[h]]$ is a composition of a multiplication

$$\phi_g : f_h \rightarrow gf_h, \quad f_h \in C^\infty(X)[[h]], \quad (3.54)$$

by an invertible element $g \in C^\infty(X)$ and an automorphism ϕ so that ϕ_0 (3.53) is the identity map. The latter takes the following form (3.52):

$$\phi(f) = (1 + h\sigma)f, \quad f \in C^\infty(X), \quad (3.55)$$

where σ is an arbitrary $\mathbb{R}[[h]]$ -linear, but not $C^\infty(X)$ -linear endomorphism of $C^\infty(X)[[h]]$. The inverse of ϕ (3.55) is

$$\phi^{-1}(f) = \left(1 + \sum_k (-1)^k h^k \circ^k \sigma \right) f,$$

where \circ denotes the composition of morphisms. Thus, a generic automorphism of $C^\infty(X)[[h]]$ is given by the expression

$$\phi(f_h) = g(1 + h\sigma)f_h, \quad f_h \in C^\infty(X)[[h]], \quad (3.56)$$

where g is an invertible element of $C^\infty(X)$. Note that it also is a $C^\infty(X)$ -linear morphism of $C^\infty(X)[[h]]$.

We say that the $\mathbb{R}[[h]]$ -linear morphism ϕ (3.51) is a differential operator if it is a finite order linear differential operator on a $\mathbb{R}[[h]]$ -module $C^\infty(X)[[h]]$. In this case, each morphism ϕ_k in the expression (3.52) is a finite order $C^\infty(X)$ -valued differential operator on $C^\infty(X)$.

In accordance with Definition 3.2, a differential operator ϕ_r is represented by a smooth real function on some finite order jet manifold $J^k Y$ or its pull-back onto the infinite order jet manifold $J^\infty F$. Then one can regard the morphism as $\mathbb{R}[[h]]$ -valued function on $J^\infty F$ whose components ϕ_r are the above-mentioned pull-back of functions on finite order jet manifolds.

In view of Theorem 3.2, we then can define a $C^\infty(X)[[h]]$ -valued m -linear differential operator ε on $C^\infty(X)[[h]]$ as follows.

DEFINITION 3.4: A $C^\infty(X)[[h]]$ -valued m -multidifferential operator ε on an $\mathbb{R}[[h]]$ -ring $C^\infty(X)[[h]]$ is an $\mathbb{R}[[h]]$ -multilinear map whose restriction on a real ring $C^\infty(X)$ is represented by an $\mathbb{R}[[h]]$ -valued m -linear function

$$\varepsilon = \sum_{r \in \mathbb{N}} h^r \varepsilon_r \quad (3.57)$$

on the product Π^m (3.27) such that its components ε_r are smooth real functions on some finite order jet manifold (3.25). \square

It follows from this definition that, in particular, any multidifferential operator \mathcal{E} on $C^\infty(X)$ (Theorem 3.2) defines the multidifferential operator (3.57) on $C^\infty(X)[[h]]$ where $\mathcal{E}_{r>0} = 0$.

For instance, the multiplication (3.40) in $C^\infty(X)$ also is a multiplication in $C^\infty(X)[[h]]$. At the same time, any bidifferential operator

$$\varepsilon(t, t') = \sum_{r \in \mathbb{N}} h^r \varepsilon_r(x, t_\Lambda, t'_\Sigma) = \sum_{r \in \mathbb{N}} h^r \varepsilon_r^{\Lambda, \Sigma}(x) t_\Lambda t'_\Sigma \quad (3.58)$$

defines a different multiplication

$$t * t' = \sum_{r \in \mathbb{N}} h^r \varepsilon_r(x, t_\Lambda, t'_\Sigma) \quad (3.59)$$

in $C^\infty(X)[[h]]$.

Following the standard terminology, we say that the multiplication (3.59) is a deformation of a ring $C^\infty(X)$. In accordance with Definition 2.2, two deformations ε and ε' of $C^\infty(X)$ are equivalent if there exists the automorphism ϕ (3.51) of $C^\infty(X)[[h]]$ so that the relation (2.3):

$$\phi(f_h *' f'_h) = \phi(f_h) * \phi(f'_h), \quad f_h, f'_h \in C^\infty(X)[[h]], \quad (3.60)$$

holds.

Now let us require that the deformation (3.59) is a ring, i.e., $C^\infty(X)[[h]]$ is a unital associative algebra with respect to this product. We prove item (i) of Lemma 2.1.

Let $\mathbf{1}_h$ be the unit element with respect to the product (3.59). Referring to the expression (3.59), we obtain

$$f = \mathbf{1}_h * f = \sum_{r \in \mathbb{N}} h^r \varepsilon_r(x, \mathbf{1}_h, f) = \sum_{r \in \mathbb{N}} h^r \varepsilon_r(x) \mathbf{1}_h f,$$

for any $f \in C^\infty(X)$. It follows that

$$\sum_{r \in \mathbb{N}} h^r \varepsilon_r(x) \mathbf{1}_h = 1,$$

that is, $\mathbf{1}_h$ is some invertible element of $C^\infty(X)[[h]]$. Then there is the automorphism $f \rightarrow (\mathbf{1}_h)^{-1} f$ (3.56) which bring $\mathbf{1}_h$ into 1.

Hereafter, we therefore assume that the deformation (3.59) preserves 1 as the unit. In this case, the coefficient function $\varepsilon_r^{\Lambda, \Sigma}$, $|\Lambda| \leq 0$, $|\Sigma| \leq 0$, in the expression (3.58) obey the relations

$$\varepsilon_0^{\cdot\cdot\cdot}(x) = 1, \quad \varepsilon_{r>0}^{\cdot\cdot\cdot\Sigma}(x) = \varepsilon_{r>0}^{\Lambda\cdot\cdot} = 0, \quad (3.61)$$

i.e., the bidifferential operators $\varepsilon_{r>0}$ do not contain zero-order differential operators. Such a deformation takes a form

$$\begin{aligned} t * t' &= tt' + \sum_{0 < r, 0 < |\Lambda|, 0 < |\Sigma|} h^r \varepsilon_r(x, t_\Lambda, t'_\Sigma) = \\ &= tt' + \sum_{0 < r, 0 < |\Lambda|, 0 < |\Sigma|} h^r \varepsilon_r^{\Lambda, \Sigma}(x) t_\Lambda t'_\Sigma. \end{aligned} \quad (3.62)$$

We also restrict our consideration to automorphisms (3.55) of $C^\infty(X)[[h]]$. Since we deal with differential deformations, such an automorphism is represented by a linear bundle morphism

$$\begin{aligned} \phi : J^\infty F &\xrightarrow{X} F, \\ t' &= t + h\phi(x, t_\Lambda) = t + h \sum_{k \in \mathbb{N}} h^k \sigma_k(x, t_\Lambda), \quad 0 \leq |\Lambda|, \end{aligned} \quad (3.63)$$

where σ_k are functions on $J^\infty F$ (i.e., linear differential operators on $C^\infty(X)$) of finite jet order which are independent of t and linear in t_Λ , $0 < |\Lambda|$. In this case, the condition (3.60) of an equivalence of different deformations ε and ε' (3.62) takes a form

$$\begin{aligned} &tt' + h(t\sigma(x, t'_\Lambda) + \sigma(x, t_\Lambda)t') + \\ &\sum_{0 < r, 0 < |\Lambda|, 0 < |\Sigma|} h^r \varepsilon_r(x, d_\Lambda(t + h\sigma(x, t_\Sigma)), d_\Sigma(t' + h\sigma(x, t'_\Sigma))) = \\ &\phi(tt') + \phi \left(\sum_{0 < r, 0 < |\Lambda|, 0 < |\Sigma|} h^r \varepsilon'_r(x, t_\Lambda, t'_\Sigma) \right). \end{aligned} \quad (3.64)$$

In particular,

$$t\sigma_0(x, t'_\Lambda) + \sigma_0(x, t_\Lambda)t' + \varepsilon_1(x, t, t') = \sigma_0(x, tt') + \varepsilon'_1(x, t, t'). \quad (3.65)$$

For instance, a deformation ε' is trivial only if

$$\varepsilon'_1(x, t, t,) = t\sigma_0(x, t'_\Lambda) + \sigma_0(x, t_\Lambda)t' - \sigma_0(x, tt'). \quad (3.66)$$

Let us now investigate the associativity condition (2.7) of the deformation (3.62):

$$(t * t') * t'' - t * (t' * t'') = 0. \quad (3.67)$$

We come to the following relation of three-linear differential operators

$$D_r(x, t, t', t'') = \sum_{0 \leq k \leq r} (\varepsilon_{r-k}(x, \varepsilon_k(t, t'), t'') - \varepsilon_{r-k}(x, t, \varepsilon_k(t', t''))) = 0. \quad (3.68)$$

In particular,

$$D_0(x, t, t', t'') = (tt')t'' - t(t't'') = 0,$$

and

$$\begin{aligned} D_1(x, t, t', t'') &= d_m \varepsilon_1(x, t, t', t'') = \\ &\varepsilon_1(x, tt', t'') - \varepsilon_1(x, t, t't'') + \varepsilon_1(x, t, t')t'' - t\varepsilon_1(x, t', t'') = 0. \end{aligned} \quad (3.69)$$

We obtain from the relation (3.69) that

$$\varepsilon_1^{\Lambda+\Lambda', \Sigma} - \varepsilon_1^{\Lambda, \Lambda'+\Sigma} = 0,$$

where functions ε_1 also obey the conditions (3.61).

Following the procedure in Section 2.1.2, let us write

$$D_r(x, t, t', t'') = E_r(x, t, t', t'') - (\delta^2 \varepsilon_r)(x, t, t', t''), \quad (3.70)$$

$$E_r(x, t, t', t'') = \quad (3.71)$$

$$\begin{aligned} & \sum_{0 < k < r} (\varepsilon_{r-k}(x, \varepsilon_k(t, t'), t'') - \varepsilon_{r-k}(x, t, \varepsilon_k(t', t''))), \\ (\delta^2 \varepsilon_r)(x, t, t', t'') &= t \varepsilon_r(x, t', t'') - \varepsilon_r(x, tt', t'') + \\ & \varepsilon_r(x, t, t' t'') - \varepsilon_r(x, t, t') t'', \end{aligned} \quad (3.72)$$

where δ^2 (3.72) is the second Hochschild coboundary operator (3.31).

For instance, a glance at the expression (3.69) shows that

$$\begin{aligned} E_1(x, t, t', t'') &= 0, \\ D_1(x, t, t', t'') &= -(\delta^2 \varepsilon_1)(x, t, t', t'') = 0, \end{aligned} \quad (3.73)$$

that is, a deformation ε is associative only if its term ε_1 is a Hochschild two-cocycle. In particular, the condition (3.73) is satisfied if ε_1 is a Hochschild coboundary, i.e.,

$$\varepsilon_1(x, t, t') = (\delta^1 \gamma)(x, t, t') = t \gamma(x, t') - \gamma(x, tt') + \gamma(x, t) t' \quad (3.74)$$

where δ^1 is the first Hochschild coboundary operator (3.30). A glance at the relation (3.66) shows that this is the case of a trivial deformation when ϕ is some automorphism (3.63) and $\gamma = \sigma_0$ (Theorem 2.2).

For instance, let γ be a first order differential operator. Since it must not contain zero order differential operator, it is a derivation, and then ε_1 (3.74) equals zero.

Let σ_0 be a second order differential operator, i.e.,

$$\gamma(x, t) = \gamma^{\mu\nu}(x) t_{\mu\nu}, \quad (3.75)$$

where certainly $\gamma^{\mu\nu} = \gamma^{\nu\mu}$. Then

$$\varepsilon_1(x, t, t') = t \gamma^{\mu\nu}(x) d_{\mu\nu} t' + t' \gamma^{\mu\nu}(x) d_{\mu\nu} t - \gamma^{\mu\nu}(x) d_{\mu\nu}(tt') = \gamma^{\mu\nu} t_\mu t_\nu \quad (3.76)$$

is a Hochschild two-coboundary. Let us consider

$$\varepsilon_1(x, t, t') = \varepsilon^{\mu\nu}(x) t_\mu t_\nu, \quad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}. \quad (3.77)$$

Then it is readily observed that $\delta^2 \varepsilon_1(x, t, t') = 0$, i.e., ε_1 (3.77) is a Hochschild two-cocycle, but not a coboundary.

Thus, we come to Theorem 2.2 where $A = C^\infty(X)$ that non-equivalent deformations of a ring $C^\infty(X)$ are parameterized by elements of the Hochschild cohomology $H^2(C^\infty(X), C^\infty(X))$, and to Theorem 2.7 that

$$H^2(C^\infty(X), C^\infty(X)) = \mathcal{T}_2(X).$$

Since we are restricted to differential deformations, their equivalence classes are associated to elements of the differential Hochschild cohomology $H^2(D^*)$. In accordance with Theorems 2.16 and 2.17, we have the isomorphism (2.19):

$$H^2(D^*) = H^2(\mathcal{T}^*) = \mathcal{T}^2. \quad (3.78)$$

Given a bivector field $\varepsilon_1 \in \mathcal{T}^2$, a problem however lies in that to write the corresponding deformation (3.62) in an explicit form. If $X = \mathbb{R}^d$ is endowed with global coordinates (x^λ) and ε_1 is given by the expression (3.77) with constant coefficients $\varepsilon^{\mu\nu}$, the Weyl product provides a desired deformation. It reads

$$t * t' = \sum_{0 \leq k} \frac{h^k}{k!} \varepsilon^{\alpha_1 \beta_1} \dots \varepsilon^{\alpha_k \beta_k} t_{\alpha_1 \dots \alpha_k} t'_{\beta_1 \dots \beta_k}. \quad (3.79)$$

Let $w \in \mathcal{T}^2$ be the Poisson bivector field (3.24), and let $*$ be the corresponding deformation. Then the commutator (2.15):

$$[f, f']_h = \frac{1}{2h} (f * f' - f' * f) \quad (3.80)$$

defines a Lie deformation of the Poisson bracket $\{, \}$ on X (Remark 2.5). For instance, let w provide a regular Poisson structure on \mathbb{R}^d and (q^i, p_i, z^a) the corresponding Darboux coordinates such that

$$w = \frac{1}{2} (t^i t'_i - t_i t'^i). \quad (3.81)$$

Then the corresponding Lie deformation is given by the Moyal product (2.21):

$$[f, f']_h = \sum_{k=0} \frac{h^k}{k!} \sum_{r=0}^k (-1)^r (t_{i_1 \dots i_r} t'^{i_{r+1} \dots i_k} t'^{i_1 \dots i_r} t_{i_{r+1} \dots i_k}). \quad (3.82)$$

Now we aim to generalize the Weyl expression (3.79) to an arbitrary bivector field (3.77) on an arbitrary manifold. In contrast to Fedosov's quantization on a symplectic manifold in a presence of a particular symplectic connection, we use the fact that that an infinite order jet manifold $J^\infty Y$ of a fibre bundle $Y \rightarrow X$ admits the canonical flat connection d_H (4.55). A necessary step to this goal is a definition the infinite order jet prolongation of differential operators.

3.3 Jet prolongation of multidifferential operators

In order to describe infinite jet order prolongation of multidifferential operators, let us start with that of differential operators.

Let $Y \rightarrow X$ and $E \rightarrow X$ be fibre bundles provided with bundle coordinates (x^λ, y^i) and (x^λ, v^A) , respectively. By virtue of Definition 4.2, a k -order E -valued differential operator on Y is the fibre bundle morphism (4.45):

$$\begin{aligned} \Delta : J^k Y &\xrightarrow{X} E, \\ v^A \circ \Delta &= \Delta^A(x^\lambda, y^i, y_\lambda^i, \dots, y_{\lambda_k \dots \lambda_1}^i). \end{aligned} \quad (3.83)$$

This morphism admits the canonical r -order jet prolongation (4.40) to a fibre bundle morphism

$$J^r \Delta : J^r J^k Y \xrightarrow{X} J^r E, \quad (3.84)$$

where we follow the notation (4.34) and (4.38). Restricted to $J^{r+k} Y \subset J^r J^k Y$ (4.39), the morphism (3.84) is an $(r+k)$ -order $J^r E$ -valued differential operator

$$\begin{aligned} \mathcal{J}^r \Delta &= J^r \Delta \circ \sigma_{rk} : J^{r+k} Y \xrightarrow{X} J^r E, \\ v_\Lambda^A \circ \mathcal{J}^r \Delta &= d_\Lambda \Delta^A, \quad |\Lambda| \leq r, \end{aligned} \quad (3.85)$$

on Y in accordance with Definition 4.2.

DEFINITION 3.5: The differential operator $\mathcal{J}^r \Delta$ (3.85) is called the r -order prolongation of the differential operator Δ (3.83). \square

Let s be a section of a fibre bundle $Y \rightarrow X$. The differential operator Δ (3.83) sends it onto the section $\Delta \circ s$ (4.46) of a fibre bundle $E \rightarrow X$. Then the r -order prolongation $\mathcal{J}^r \Delta$ (3.85) of Δ (3.83) sends s onto an integral section of the jet bundle $J^r E \rightarrow X$ which is the r -order jet prolongation $J^r(\Delta \circ s)$ (4.46) of a section $\Delta \circ s$ of Y .

Let $\mathcal{J}^{r+1} \Delta$ be the $(r+1)$ -order prolongation (3.85) of the differential operator Δ (3.83). Then we have a commutative diagram

$$\begin{array}{ccccc} & & J^{r+k} Y & \xleftarrow{\pi_{r+k}^{r+k+1}} & J^{r+k} Y \\ \mathcal{J}^r \Delta & & \downarrow & & \downarrow & \mathcal{J}^{r+1} \Delta \\ & & J^r E & \xleftarrow{\pi_r^{r+1}} & J^{r+1} E \end{array} \quad (3.86)$$

Let us consider the inverse system of jet manifolds $J^k Y$ (4.47):

$$X \xleftarrow{\pi} Y \xleftarrow{\pi_0^1} \dots \xleftarrow{\pi_{k-1}^k} J^{k-1} Y \xleftarrow{\pi_{k-1}^k} J^k Y \xleftarrow{\pi_k^{k+1}} \dots, \quad (3.87)$$

and the inverse system

$$X \xleftarrow{\pi} E \xleftarrow{\pi_0^1} \dots \xleftarrow{\pi_{r-1}^r} J^{r-1} E \xleftarrow{\pi_{r-1}^r} J^r E \xleftarrow{\pi_r^{r+1}} \dots \quad (3.88)$$

of jet manifolds $J^r E$. Then the commutative diagrams (3.86) are assembled into an order preserving morphism

$$\begin{array}{ccccccc} \dots & \xleftarrow{\mathcal{J}^r \Delta} & J^{r+k} Y & \xleftarrow{\pi_{r+k}^{r+k+1}} & J^{r+k} Y & \xleftarrow{\mathcal{J}^{r+1} \Delta} & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \xleftarrow{\mathcal{J}^r \Delta} & J^r E & \xleftarrow{\pi_r^{r+1}} & J^{r+1} E & \xleftarrow{\mathcal{J}^{r+1} \Delta} & \dots \end{array} \quad (3.89)$$

of the inverse systems (3.87) and (3.88). Let $J^\infty Y$ and $J^\infty E$ be the inverse limits (4.48) of the inverse systems (3.87) and (3.88), respectively. Then there is a morphism

$$\mathcal{J}^\infty \Delta = \varprojlim \mathcal{J}^r \Delta : J^\infty Y \rightarrow J^\infty E \quad (3.90)$$

so that the diagrams

$$\begin{array}{ccccc}
& J^{r+k}Y & \xleftarrow{\pi_{r+k}^\infty} & J^\infty Y & \\
\mathcal{J}^r \Delta \swarrow & \downarrow & & \downarrow & \searrow \mathcal{J}^\infty \Delta \\
& J^r E & \xleftarrow{\pi_r^\infty} & J^\infty E &
\end{array} \tag{3.91}$$

are commutative for any $r \geq 0$.

DEFINITION 3.6: The morphism $\mathcal{J}^\infty \Delta$ (3.90) is called the infinite order prolongation of the differential operator Δ (3.83). \square

Given the coordinate atlas (x^λ, y_Λ^i) (4.50) of $J^\infty Y$ and the coordinate atlas (x^λ, v_Λ^A) (4.50) of $J^\infty E$, the morphism $\mathcal{J}^\infty \Delta$ (3.90) reads

$$v_\Lambda^A \circ \mathcal{J}^\infty \Delta = d_\Lambda \Delta^A, \quad 0 \leq |\Lambda|. \tag{3.92}$$

Now let $F = X \times \mathbb{R}$ be the fibre bundle (3.1) endowed with bundle coordinates (x^λ, t) . In accordance with Definition 3.2, a k -order $C^\infty(X)$ -valued differential operator \mathcal{E} on a real vector space $C^\infty(X)$ is defined to be a smooth real function \mathcal{E} on the jet manifold $J^k F$, i.e., a section of the pull-back bundle (3.17).

At the same time, basing on Definition 4.2 of differential operators, we come to another equivalent definition of \mathcal{E} .

DEFINITION 3.7: A k -order $C^\infty(X)$ -valued differential operator on a real vector space $C^\infty(X)$ is a fibre bundle morphism

$$\begin{aligned}
\mathcal{E} : J^k F &\xrightarrow[X]{} F, \\
t' \circ \mathcal{E} &= \mathcal{E}(x^\lambda, t, t_\lambda, \dots, t_{\lambda_k \dots \lambda_1}).
\end{aligned} \tag{3.93}$$

\square

In accordance with Definition 3.5, an r -order prolongation of the differential operator \mathcal{E} (3.93) is the morphism

$$\begin{aligned}
\mathcal{J}^r \mathcal{E} : J^{r+k} F &\xrightarrow[X]{} J^r F, \\
t'_\Lambda \circ \mathcal{J}^r \mathcal{E} &= d_\Lambda \mathcal{E}, \quad |\Lambda| \leq r.
\end{aligned} \tag{3.94}$$

The differential operator $\mathcal{J}^r \mathcal{E}$ sends a section f of the fibre bundle (3.1) (i.e., a smooth real function f on X) onto an integral section of the jet bundle $J^r F \rightarrow X$ which is the r -order jet prolongation $J^r(\mathcal{E} \circ f)$ (4.46) of a function $\mathcal{E} \circ f$.

Let $\mathcal{J}^{r+1} \mathcal{E}$ be the $(r+1)$ -order prolongation (3.94) of the differential operator \mathcal{E} (3.93). Then we have the commutative diagram (3.86):

$$\begin{array}{ccccc}
& J^{r+k} F & \xleftarrow{\pi_{r+k}^{r+k+1}} & J^{r+k} F & \\
\mathcal{J}^r \mathcal{E} \swarrow & \downarrow & & \downarrow & \searrow \mathcal{J}^{r+1} \mathcal{E} \\
& J^r F & \xleftarrow{\pi_r^{r+1}} & J^{r+1} F &
\end{array}$$

These diagrams are assembled into an endomorphism

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & J^{r+k}F & \xleftarrow{\pi_{r+k}^{r+k+1}} & J^{r+k}F & \longleftarrow & \cdots \\
& & \downarrow \mathcal{J}^r \mathcal{E} & & \downarrow \mathcal{J}^{r+1} \mathcal{E} & & \\
\cdots & \longleftarrow & J^r F & \xleftarrow{\pi_r^{r+1}} & J^{r+1} F & \longleftarrow & \cdots
\end{array} \quad (3.95)$$

of the inverse system (3.13). This endomorphism (3.95) of the inverse system (3.13) yields a morphism

$$\begin{aligned}
\mathcal{J}^\infty \mathcal{E} &= \varprojlim \mathcal{J}^r \mathcal{E} : J^\infty F \rightarrow J^\infty F, \\
t'_\Lambda \circ \mathcal{J}^\infty \mathcal{E} &= d_\Lambda \mathcal{E}, \quad 0 \leq |\Lambda|,
\end{aligned} \quad (3.96)$$

of its inverse limit $J^\infty F$ (3.14) so that the diagrams

$$\begin{array}{ccccc}
& J^{r+k}F & \xleftarrow{\pi_{r+k}^\infty} & J^\infty F & \\
\mathcal{J}^r \mathcal{E} \downarrow & & & \downarrow & \mathcal{J}^\infty \mathcal{E} \\
& J^r F & \xleftarrow{\pi_r^\infty} & J^\infty F &
\end{array}$$

are commutative for any $r \geq 0$. Following Definition 3.6, one can think of the morphism (3.96) as being the infinite order prolongation of the differential operator (3.93).

Let f be a section of the fibre bundle $F \rightarrow X$ (3.1) (i.e., a smooth real function on X). Its r -order jet prolongation $J^r f$ (4.41) is an integrable section of the jet bundle $J^r F \rightarrow X$. There is a map

$$\begin{aligned}
J^\infty f &: X \rightarrow J^\infty F, \\
t_\Lambda \circ J^\infty f &= \partial_\Lambda f, \quad |\Lambda| \geq 0,
\end{aligned} \quad (3.97)$$

so that

$$J^r f = \pi_r^\infty \circ J^\infty f \quad (3.98)$$

for any $r \geq 0$. One calls $J^\infty f$ (3.97) the infinite order jet prolongation of a function f .

In particular, given a function f and its image $\mathcal{E} \circ f$ with respect to a differential operator \mathcal{E} , let $J^\infty(\mathcal{E} \circ f)$ be its infinite order jet prolongation. Then the morphism $\mathcal{J}^\infty \mathcal{E}$ (3.96) sends f onto $J^\infty(\mathcal{E} \circ f)$ so that the relations (3.98):

$$\mathcal{J}^r \mathcal{E} \circ f = \pi_{r+k}^\infty \circ \mathcal{J}^\infty \mathcal{E} \circ f,$$

are satisfied.

Turn now to jet prolongations of multidifferential operators. By virtue of Theorem 3.2, a $C^\infty(X)$ -valued m -multilinear k -order differential operator is defined by a m -linear smooth real function on the k -order jet manifold (3.26), i.e., it is a bundle morphism

$$\begin{aligned}
\mathcal{E} &: J^k(\times_X^m F) \rightarrow F, \\
t' &= \mathcal{E}(x^\lambda, y_\Lambda^1, \dots, y_\Lambda^m), \quad |\Lambda| \leq k,
\end{aligned} \quad (3.99)$$

(cf. Definition 3.7). Then by virtue of Definition 3.5, an r -order prolongation of the multidifferential operator \mathcal{E} (3.99) is the morphism

$$\begin{aligned} \mathcal{J}^r \mathcal{E} : J^{r+k} \left(\bigotimes_X^m F \right) &\xrightarrow{X} J^r F, \\ t'_\Lambda \circ \mathcal{J}^r \mathcal{E} &= d_\Lambda \mathcal{E}, \quad |\Lambda| \leq r. \end{aligned} \quad (3.100)$$

Accordingly, it follows from Definition 3.6 that the infinite order jet prolongation of the multidifferential operator \mathcal{E} (3.99) is the morphism

$$\begin{aligned} \mathcal{J}^\infty \mathcal{E} : J^\infty \left(\bigotimes_X^m F \right) &\rightarrow J^\infty F, \\ t'_\Lambda \circ \mathcal{J}^\infty \mathcal{E} &= d_\Lambda \mathcal{E}, \quad 0 \leq |\Lambda|. \end{aligned} \quad (3.101)$$

3.4 Star-product in a covariant form

Let X be a smooth manifold. We start with the notion of a connection on $C^\infty(X)$ -modules [35, 46].

DEFINITION 3.8: Let P be a $C^\infty(X)$ -module. An \mathbb{R} -module morphism $\nabla : Q \rightarrow P$ is called the derivation of Q if it satisfies the Leibniz rule

$$\nabla(fq) = \partial(f)p + f\nabla(p), \quad q \in Q, \quad f \in C^\infty(X),$$

where ∂ is a derivation of a real ring $C^\infty(X)$. \square

Derivations of a $C^\infty(X)$ -module constitute a $C^\infty(X)$ -module that we denote $\mathfrak{d}Q$. In particular, a derivation of a real ring $C^\infty(X)$ also is a derivation of $C^\infty(X)$ as a $C^\infty(X)$ -module. Let us recall that there is one-to-one correspondence between the vector fields τ on X and the derivations

$$\nabla_\tau(f) = \tau^\mu \partial_\mu f$$

of a real ring $C^\infty(X)$. this correspondence provides a $C^\infty(X)$ -module isomorphism $\mathfrak{d}C^\infty(X) = \mathcal{T}_1(X)$.

DEFINITION 3.9: A connection ∇ on a $C^\infty(X)$ -module Q is defined to be a $C^\infty(X)$ -module monomorphism

$$\nabla : \mathfrak{d}C^\infty(X) \rightarrow \mathfrak{d}Q \quad (3.102)$$

which associates some derivation ∇_τ of Q to each vector field τ on X . \square

Given a fibre bundle $Y \rightarrow X$, let $J^\infty Y$ (4.48) be the infinite order jet manifold and \mathcal{O}_∞^* (4.52) the $C^\infty(X)$ -algebra of exterior forms of finite jet order on X . Let us consider the total differential d_H (4.55). This is a \mathbb{R} -module morphism of \mathcal{O}_∞^* so that, for any vector field τ on X , it defines a derivation

$$\nabla_\tau(\phi) = \tau^\lambda d_\lambda \phi, \quad \phi \in \mathcal{O}_\infty^0, \quad (3.103)$$

of a $C^\infty(X)$ -module \mathcal{O}_∞^0 of smooth real functions of finite jet order on $J^\infty Y$. In accordance with Definition 3.9, d_H is a connection on a $C^\infty(X)$ -module \mathcal{O}_∞^0 .

Let now $Y = F$ be the fibre bundle (3.1) coordinated by (x^λ, t) and $J^\infty F$ its infinite order jet manifold (3.14). Let $C^\infty(J^\infty F) = \mathcal{O}_r^0$ be a $C^\infty(X)$ -module (3.16) of smooth real functions of finite jet order on $J^\infty F$ (Definition 3.1). Then the total differential d_H where d_λ are the total derivatives (3.12) defines a canonical connection on $C^\infty(J^r F)$ such that, for any vector field τ , we have the derivation (3.103):

$$\nabla_\tau(\phi) = \tau \lrcorner d_H(\phi) = \tau^\lambda d_\lambda \phi, \quad \phi \in C^\infty(J^r F), \quad (3.104)$$

of $C^\infty(J^\infty F)$.

In accordance with Theorem 3.1, a function $f \in C^\infty(J^\infty F)$ is a finite order differential operator on a real ring $C^\infty(X)$. Its derivation $\nabla_\tau(\phi)$ (3.104) also is an element of $C^\infty(J^\infty F)$ and, consequently, a differential operator of finite jet order on $C^\infty(X)$. It is represented as a composition of the following morphisms

$$J^\infty F \xrightarrow[X]{\mathcal{J}^\infty \phi} J^\infty F \xrightarrow[X]{\nabla_\tau} F \quad (3.105)$$

where $\mathcal{J}^\infty \phi$ is the infinite jet order prolongation (3.96) of a differential operator ϕ on $C^\infty(X)$ and $\nabla_\tau = \tau^\lambda t_\lambda$. Due to the isomorphism (3.6), this morphism is globally defined.

Remark 3.4: The morphism ∇_τ in the composition (3.105) requires first order jet prolongation of a differential operator ϕ . In a general setting, if we consider an r -order differential operator on $C^\infty(J^\infty F)$, they imply r -order jet prolongation (3.94) of differential operators ϕ on $C^\infty(X)$. \square

Turn now to multidifferential operators. Let us consider a product bundle

$$\times_X^m F \rightarrow X,$$

coordinated by (x^λ, t^i) , $i = 1, \dots, m$, where t^i are global coordinates possessing identity transition functions. Let Π^m be the corresponding infinite order jet manifold (3.27). Let us consider a $C^\infty(X)$ -module $C^\infty(\Pi^m)$ of smooth real functions of finite jet order on Π^m . In accordance with Theorem 3.2, elements of $C^\infty(\Pi^m)$ are $C^\infty(X)$ -valued multidifferential operators on $C^\infty(X)$. A $C^\infty(X)$ -module $C^\infty(\Pi^m)$ is endowed with the canonical connection

$$d_H = dx^\lambda \otimes d_\lambda = dx^\lambda (\partial_\lambda + \sum_{0 \leq |\Lambda|, 1 \leq i \leq m} t_{\lambda+\Lambda}^i \partial_i^\Lambda). \quad (3.106)$$

Since Π^m is the direct product, we also can consider the connections

$$d_H^i = dx^\lambda \otimes d_\lambda^i = dx^\lambda (\partial_\lambda + \sum_{0 \leq |\Lambda|} t_{\lambda+\Lambda}^i \partial_i^\Lambda) \quad (3.107)$$

for every $i = 1, \dots, m$. Given a multivector field

$$\vartheta = \frac{1}{m!} \vartheta^{\lambda_1 \dots \lambda_m}(x) \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_m}, \quad (3.108)$$

let us consider the composition of derivations

$$\widehat{\vartheta} = \frac{1}{m!} \vartheta^{\lambda_1 \dots \lambda_m}(x) d_{\lambda_1}^1 \circ \dots \circ d_{\lambda_m}^m, \quad (3.109)$$

acting on a $C^\infty(X)$ -module $C^\infty(\Pi^m)$. Owing to the isomorphism (3.6), this composition is globally defined. Let us call it a multiderivation. We also have a smooth real function

$$\vartheta = \frac{1}{m!} \vartheta^{\lambda_1 \dots \lambda_m}(x) d_{\lambda_1}^1 \circ \dots \circ d_{\lambda_m}^m, \quad (3.110)$$

Let us consider a function $\vartheta_0 = t^1 \dots t^m$ on Π^m and its multiderivation

$$\widehat{\vartheta}(\vartheta_0) = \frac{1}{m!} \vartheta^{\lambda_1 \dots \lambda_m}(x) t_{\lambda_1}^1 \circ \dots \circ t_{\lambda_m}^m. \quad (3.111)$$

This is a smooth real function on Π^m which provides a multilinear differential operator on $C^\infty(X)$. One also construct compositions of the multiderivations (3.110) that however implies the corresponding jet prolongation of the operator ϑ_0 (Remark 3.4).

In particular, let consider the product $F \times F$ coordinated by (x^λ, t, t') , and let $C^\infty(\Pi^2)$ be a $C^\infty(X)$ -module of smooth real functions of finite jet order on Π^2 . It is endowed with the canonical connections d_H^t and $d_H^{t'}$ (3.107). Let

$$\varepsilon = \frac{1}{2} w^{\alpha\beta}(x) \partial_\alpha \wedge \partial_\beta$$

be a bivector field on X . It defines the corresponding biderivation

$$\widehat{\varepsilon} = \varepsilon^{\lambda_1 \dots \lambda_m}(x) d_{\lambda_1}^t \circ \dots \circ d_{\lambda_m}^{t'}, \quad (3.112)$$

of a $C^\infty(X)$ -module $C^\infty(\Pi^2)$. Given a deformation parameter h , let us consider the composition of this biderivation written in the compact form

$$\exp[h\widehat{\varepsilon}] = \sum_{0 \leq k} \frac{h^k}{k!} \circ^k \widehat{\varepsilon} \dots \widehat{\varepsilon}. \quad (3.113)$$

Let this derivation act on a function $\varepsilon_0 = tt' \in C^\infty(\Pi^2)$. Then the result

$$t * t' = \exp[h\widehat{\varepsilon}](tt') = tt' + \frac{1}{2} \varepsilon(x) \alpha \beta t_\alpha t'_\beta + \quad (3.114)$$

$$\sum_{1 \leq k} \frac{h^k}{k!} \circ^k [\varepsilon^{\alpha_1 \beta_1} d_{\alpha_1} d'_{\beta_1}] \circ \dots \circ [\varepsilon^{\alpha_k \beta_k} d_{\alpha_k} d'_{\beta_k}](\varepsilon(x) \alpha \beta t_\alpha t'_\beta)$$

is a differential operator on $C^\infty(X)$. It can be treated as a star-product.

A direct computation shows that the star-product obeys the associativity condition.

4 Appendix

This Section summarizes the relevant material on fibre bundles and jet manifolds [21, 35, 47, 48].

All morphisms are smooth (i.e. of class C^∞) and manifolds are smooth real and finite-dimensional. A smooth manifold is conventionally assumed to be Hausdorff and second-countable. Consequently, it is locally compact and paracompact. Being paracompact, a smooth manifold admits a partition of unity by smooth real functions. Unless otherwise stated, manifolds are assumed to be connected.

The standard symbols \otimes , \vee , \wedge and \lrcorner stand for the tensor, symmetric, exterior and interior products, respectively. By $C^\infty(Z)$ is denoted the ring of smooth real functions on a manifold Z .

If Z is a manifold Z , we denote by

$$\pi_Z : TZ \rightarrow Z, \quad \pi_Z^* : T^*Z \rightarrow Z$$

its tangent and cotangent bundles, respectively. Given coordinates (z^α) on Z , they are equipped with the holonomic coordinates

$$(z^\lambda, \dot{z}^\lambda), \quad \dot{z}'^\lambda = \frac{\partial z'^\lambda}{\partial z^\mu} \dot{z}^\mu, \quad (4.1)$$

$$(z^\lambda, \dot{z}_\lambda), \quad \dot{z}'_\lambda = \frac{\partial z'^\mu}{\partial z^\lambda} \dot{z}_\mu, \quad (4.2)$$

with respect to the holonomic frames $\{\partial_\lambda\}$ and coframes $\{dz^\lambda\}$ in the tangent and cotangent spaces to Z , respectively. Any manifold morphism $f : Z \rightarrow Z'$ yields the tangent morphism

$$Tf : TZ \rightarrow TZ', \quad \dot{z}'^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial x^\mu} \dot{z}^\mu. \quad (4.3)$$

4.1 Fibre bundles

Let us consider manifold morphisms of maximal rank. They are immersions and submersions. An injective immersion is a submanifold. If an immersion f is a homeomorphism onto $f(M)$ equipped with the relative topology from N , it is called the imbedding. If $M \subset N$, its natural injection is denoted by $i_M : M \rightarrow N$.

A surjective submersion

$$\pi : Y \rightarrow X, \quad \dim X = n > 0, \quad (4.4)$$

is called the fibred manifold, i.e., the tangent morphism $T\pi : TY \rightarrow TX$ is a surjection. One says that Y is a total space of the fibred manifold (4.4), X is its base, π is a fibration, and $Y_x = \pi^{-1}(x)$ is a fibre over $x \in X$.

THEOREM 4.1: A surjection $Y \rightarrow X$ is a fibred manifold iff a manifold Y admits an atlas of fibred coordinate charts $(U_Y; x^\lambda, y^i)$ such that (x^λ) are coordinates

on $\pi(U_Y) \subset X$ and coordinate transition functions read $x'^\lambda = f^\lambda(x^\mu)$, $y'^i = f^i(x^\mu, y^j)$. \square

The fibred manifold (4.4) admits a local section. This is an injection $s : U \rightarrow Y$ of an open subset $U \subset X$ such that $\pi \circ s = \text{Id } U$. If $U = X$, one calls s the global section. A global section need not exist.

THEOREM 4.2: A fibred manifold whose fibres are diffeomorphic to \mathbb{R}^m always has a global section. \square

The fibred manifold $Y \rightarrow X$ (4.4) is called the fibre bundle if admits a fibred coordinate atlas $\{(\pi^{-1}(U_\xi); x^\lambda, y^i)\}$ over a cover $\{\pi^{-1}(U_i)\}$ of Y which is the inverse image of a cover $\{U_\xi\}$ of X . In this case, there exists a manifold V , called the typical fibre, such that Y is locally diffeomorphic to the splittings

$$\psi_\xi : \pi^{-1}(U_\xi) \rightarrow U_\xi \times V, \quad (4.5)$$

glued together by means of transition functions

$$\varrho_{\xi\zeta} = \psi_\xi \circ \psi_\zeta^{-1} : U_\xi \cap U_\zeta \times V \rightarrow U_\xi \cap U_\zeta \times V \quad (4.6)$$

on overlaps $U_\xi \cap U_\zeta$. Restricted to a point $x \in X$, trivialization morphisms ψ_ξ (4.5) and transition functions $\varrho_{\xi\zeta}$ (4.6) define diffeomorphisms of fibres

$$\psi_\xi(x) : Y_x \rightarrow V, \quad x \in U_\xi, \quad (4.7)$$

$$\varrho_{\xi\zeta}(x) : V \rightarrow V, \quad x \in U_\xi \cap U_\zeta. \quad (4.8)$$

Trivialization charts (U_ξ, ψ_ξ) together with transition functions $\varrho_{\xi\zeta}$ (4.6) constitute a bundle atlas

$$\Psi = \{(U_\xi, \psi_\xi), \varrho_{\xi\zeta}\} \quad (4.9)$$

of a fibre bundle $Y \rightarrow X$. Two bundle atlases are said to be equivalent if their union also is a bundle atlas, i.e., there exist transition functions between trivialization charts of different atlases. A fibre bundle $Y \rightarrow X$ is uniquely defined by a bundle atlas.

Without a loss of generality, we assume that a cover for a bundle atlas of $Y \rightarrow X$ also is a cover for a manifold atlas of a base X . Then, given the bundle atlas Ψ (4.9), a fibre bundle $Y \rightarrow X$ is provided with the associated bundle coordinates

$$x^\lambda(y) = (x^\lambda \circ \pi)(y), \quad y^i(y) = (y^i \circ \psi_\xi)(y), \quad y \in \pi^{-1}(U_\xi),$$

where x^λ are coordinates on $U_\xi \subset X$ and y^i , called fibre coordinates, are coordinates on a typical fibre V .

A fibre bundle $Y \rightarrow X$ is said to be trivial if Y is diffeomorphic to a product $X \times V$.

THEOREM 4.3: Any fibre bundle over a contractible base is trivial. \square

A bundle morphism of a fibre bundle $\pi : Y \rightarrow X$ to a fibre bundle $\pi' : Y' \rightarrow X'$, by definition, is a fibrewise morphism. It is defined as a pair (Φ, f) of manifold morphisms which form a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{f} & X' \end{array}, \quad \pi' \circ \Phi = f \circ \pi.$$

Bundle injections and surjections are called bundle monomorphisms and epimorphisms, respectively. A bundle diffeomorphism is called a bundle isomorphism, or a bundle automorphism if it is an isomorphism to itself. For the sake of brevity, a bundle morphism over $f = \text{Id } X$ is said to be a bundle morphism over X , and is denoted by $Y \xrightarrow{X} Y'$. In particular, a bundle automorphism over X is called a vertical automorphism.

A bundle monomorphism $\Phi : Y \rightarrow Y'$ over X is called a subbundle of a fibre bundle $Y' \rightarrow X$ if $\Phi(Y)$ is a submanifold of Y' .

Let us mention the following standard constructions of new fibre bundles from the old ones.

- Given a fibre bundle $\pi : Y \rightarrow X$ and a manifold morphism $f : X' \rightarrow X$, the pull-back of Y by f is called the manifold

$$f^*Y = \{(x', y) \in X' \times Y : \pi(y) = f(x')\} \quad (4.10)$$

together with a natural projection $(x', y) \rightarrow x'$. It is a fibre bundle over X' such that the fibre of f^*Y over a point $x' \in X'$ is that of Y over a point $f(x') \in X$. There is the canonical bundle morphism

$$f_Y : f^*Y \ni (x', y)|_{\pi(y)=f(x')} \xrightarrow{f} y \in Y. \quad (4.11)$$

Any section s of a fibre bundle $Y \rightarrow X$ yields the pull-back section $f^*s(x') = (x', s(f(x')))$ of $f^*Y \rightarrow X'$.

- If $X' \subset X$ is a submanifold of X and $i_{X'}$ is the corresponding natural injection, then the pull-back bundle $i_{X'}^*Y = Y|_{X'}$ is called the restriction of a fibre bundle Y to a submanifold $X' \subset X$. If X' is an imbedded submanifold, any section of the pull-back bundle $Y|_{X'} \rightarrow X'$ is the restriction to X' of some section of $Y \rightarrow X$.

- Let $\pi : Y \rightarrow X$ and $\pi' : Y' \rightarrow X$ be fibre bundles over a base X . Their bundle product $Y \times_X Y'$ over X is defined as the pull-back

$$Y \times_X Y' = \pi^*Y' \quad \text{or} \quad Y \times_X Y' = \pi'^*Y$$

together with its natural surjection onto X . Fibres of the bundle product $Y \times_X Y'$ are the Cartesian products $Y_x \times Y'_x$ of fibres of fibre bundles Y and Y' .

A vector bundle is a fibre bundle $Y \rightarrow X$ such that:

- its typical fibre V and all fibres $Y_x = \pi^{-1}(x)$, $x \in X$, are real finite-dimensional vector spaces;
- there is the bundle atlas Ψ (4.9) of $Y \rightarrow X$ whose trivialization morphisms ψ_ξ (4.7) and transition functions $\varrho_{\xi\zeta}$ (4.8) are linear isomorphisms.

Accordingly, a vector bundle is provided with linear bundle coordinates (y^i) possessing linear transition functions $y'^i = A_j^i(x)y^j$. We have

$$y = y^i e_i(\pi(y)) = y^i \psi_\xi(\pi(y))^{-1}(e_i), \quad \pi(y) \in U_\xi, \quad (4.12)$$

where $\{e_i\}$ is a fixed basis for the typical fibre V of Y , and $\{e_i(x)\}$ are the fibre bases (or the frames) for fibres Y_x of Y associated to a bundle atlas Ψ .

By virtue of Theorem 4.2, any vector bundle has a global section, e.g., the canonical global zero-valued section $\hat{0}(x) = 0$. Global sections of a vector bundle $Y \rightarrow X$ constitute a $C^\infty(X)$ -module $Y(X)$.

By a morphism of vector bundles is meant a linear bundle morphism, whose restriction to each fibre is a linear map.

There are the following particular constructions of new vector bundles from the old ones.

- Let $Y \rightarrow X$ be a vector bundle with a typical fibre V . By $Y^* \rightarrow X$ is denoted the dual vector bundle with the typical fibre V^* dual of V . The interior product of Y and Y^* is defined as a fibred morphism

$$\rfloor : Y \otimes Y^* \xrightarrow{X} X \times \mathbb{R}.$$

- Let $Y \rightarrow X$ and $Y' \rightarrow X$ be vector bundles with typical fibres V and V' , respectively. Their Whitney sum $Y \oplus_X Y'$ is a vector bundle over X with the typical fibre $V \oplus V'$.

- Given vector bundles Y and Y' over the same base X , their tensor product $Y \otimes Y'$ is a vector bundle over X whose fibres are the tensor products of fibres of Y and Y' . Similarly, the exterior product $Y \wedge Y$ is defined. Accordingly,

$$\wedge Y = (X \times \mathbb{R}) \oplus_X Y \oplus_X \overset{2}{\wedge} Y \oplus_X \cdots \oplus_X^m \wedge Y, \quad m = \dim V, \quad (4.13)$$

is called the exterior bundle of Y .

The tangent bundle TZ and the cotangent bundle T^*Z of a manifold Z exemplify vector bundles. Given an atlas $\Psi_Z = \{(U_\iota, \phi_\iota)\}$ of a manifold Z , the tangent bundle is provided with the holonomic bundle atlas $\Psi_T = \{(U_\iota, \psi_\iota = T\phi_\iota)\}$, where $T\phi_\iota$ is the tangent morphism to ϕ_ι . The associated linear bundle coordinates are holonomic coordinates (z^λ) with respect to the holonomic frames $\{\partial_\lambda\}$ in tangent spaces $T_z Z$.

A tensor product of tangent and cotangent bundles

$$T = (\overset{m}{\otimes} TZ) \otimes (\overset{k}{\otimes} T^*Z), \quad m, k \in \mathbb{N}, \quad (4.14)$$

is called a tensor bundle, provided with holonomic bundle coordinates $\dot{x}_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_m}$ possessing transition functions

$$\dot{x}_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_m} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\alpha_m}}{\partial x^{\mu_m}} \frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\nu_k}}{\partial x'^{\beta_k}} \dot{x}_{\nu_1 \dots \nu_k}^{\mu_1 \dots \mu_m}.$$

Let $\pi_Y : TY \rightarrow Y$ be the tangent bundle of a fibre bundle $\pi : Y \rightarrow X$. Given bundle coordinates (x^λ, y^i) on Y , it is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$. The tangent bundle $TY \rightarrow Y$ has a subbundle $VY = \text{Ker}(T\pi)$, which consists of the vectors tangent to fibres of Y . It is called the vertical tangent bundle of Y , and it is provided with the holonomic coordinates $(x^\lambda, y^i, \dot{y}^i)$ with respect to the vertical frames $\{\partial_i\}$. Every bundle morphism $\Phi : Y \rightarrow Y'$ yields a linear bundle morphism over Φ of the vertical tangent bundles

$$V\Phi : VY \rightarrow VY', \quad \dot{y}^i \circ V\Phi = \frac{\partial \Phi^i}{\partial y^j} \dot{y}^j. \quad (4.15)$$

It is called the vertical tangent morphism.

4.2 Differential forms and multivector fields

Vector fields on a manifold Z are global sections of the tangent bundle $TZ \rightarrow Z$. They are assembled into a real Lie algebra $\mathcal{T}_1(Z)$ with respect to the Lie bracket $[\cdot, \cdot]$.

A vector field u on a fibre bundle $Y \rightarrow X$ is called projectable if it projects onto a vector field on X , i.e., there exists a vector field τ on X such that $\tau \circ \pi = T\pi \circ u$. Projectable vector fields take a coordinate form

$$u = u^\lambda(x^\mu) \partial_\lambda + u^i(x^\mu, y^j) \partial_i, \quad \tau = u^\lambda \partial_\lambda.$$

A projectable vector field is called vertical if its projection onto X vanishes, i.e., it lives in the vertical tangent bundle VY .

An exterior r -form on a manifold Z is a section

$$\phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}$$

of the exterior product $\wedge^r T^*Z \rightarrow Z$. Let $\mathcal{O}^r(Z)$ denote the vector space of exterior r -forms on a manifold Z . By definition, $\mathcal{O}^0(Z) = C^\infty(Z)$ is the ring of smooth real functions on Z . All exterior forms on Z constitute the \mathbb{N} -graded commutative algebra $\mathcal{O}^*(Z)$ of global sections of the exterior bundle $\wedge T^*Z$ (4.13). This algebra is provided with the exterior differential

$$d\phi = dz^\mu \wedge \partial_\mu \phi = \frac{1}{r!} \partial_\mu \phi_{\lambda_1 \dots \lambda_r} dz^\mu \wedge dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}.$$

This obeys the relations

$$d \circ d = 0, \quad d(\phi \wedge \sigma) = d(\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge d(\sigma),$$

where the symbol $|\phi|$ stands for the form degree.

Given a manifold morphism $f : Z \rightarrow Z'$, any exterior k -form ϕ on Z' yields the pull-back exterior form $f^*\phi$ on Z given by the condition

$$f^*\phi(v^1, \dots, v^k)(z) = \phi(Tf(v^1), \dots, Tf(v^k))(f(z))$$

for an arbitrary collection of tangent vectors $v^1, \dots, v^k \in T_z Z$. The relations

$$f^*(\phi \wedge \sigma) = f^*\phi \wedge f^*\sigma, \quad df^*\phi = f^*(d\phi)$$

hold. In particular, given a fibre bundle $\pi : Y \rightarrow X$, the pull-back onto Y of exterior forms on X by π provides the monomorphism of graded commutative algebras $\mathcal{O}^*(X) \rightarrow \mathcal{O}^*(Y)$. Elements of its range $\pi^*\mathcal{O}^*(X)$ are called basic forms. Exterior forms on Y such that $u \lrcorner \phi = 0$ for an arbitrary vertical vector field u on Y are said to be horizontal forms.

The interior product (or contraction) of a vector field u and an exterior r -form ϕ on a manifold Z is given by the coordinate expression

$$u \lrcorner \phi = \frac{1}{(r-1)!} u^\mu \phi_{\mu\lambda_2 \dots \lambda_r} dz^{\lambda_2} \wedge \dots \wedge dz^{\lambda_r}. \quad (4.16)$$

It obeys the relations

$$\phi(u_1, \dots, u_r) = u_r \lrcorner \dots \lrcorner u_1 \lrcorner \phi, \quad u \lrcorner (\phi \wedge \sigma) = u \lrcorner \phi \wedge \sigma + (-1)^{|\phi|} \phi \wedge u \lrcorner \sigma.$$

The Lie derivative of an exterior form ϕ along a vector field u is

$$\mathbf{L}_u \phi = u \lrcorner d\phi + d(u \lrcorner \phi), \quad \mathbf{L}_u(\phi \wedge \sigma) = \mathbf{L}_u \phi \wedge \sigma + \phi \wedge \mathbf{L}_u \sigma.$$

In particular, if f is a function, then $\mathbf{L}_u f = u(f) = u \lrcorner df$.

Similarly to the graded commutative algebra of differential forms, the graded commutative algebra of multivector fields on a manifold Z is introduced. A multivector field ϑ of degree $|\vartheta| = r$ (or, simply, an r -vector field) on a manifold Z is a section

$$\vartheta = \frac{1}{r!} \vartheta^{\lambda_1 \dots \lambda_r} \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_r} \quad (4.17)$$

of the exterior product $\wedge^r TZ \rightarrow Z$. Let $\mathcal{T}_r(Z)$ denote the vector space of r -vector fields on Z . By definition, $\mathcal{T}_0(Z) = C^\infty(Z)$. All multivector fields on a manifold Z make up the \mathbb{N} -graded commutative algebra $\mathcal{T}_*(Z)$ of global sections of the exterior bundle $\wedge^r TZ$ (4.13) with respect to the exterior product \wedge .

The graded commutative algebra $\mathcal{T}_*(Z)$ is endowed with the Schouten–Nijenhuis bracket

$$\begin{aligned} [\cdot, \cdot]_{\text{SN}} : \mathcal{T}_r(M) \times \mathcal{T}_s(M) &\rightarrow \mathcal{T}_{r+s-1}(M), \\ [\vartheta, v]_{\text{SN}} &= \vartheta \bullet v + (-1)^{rs} v \bullet \vartheta, \\ \vartheta \bullet v &= \frac{r}{r!s!} (\vartheta^{\mu\lambda_2 \dots \lambda_r} \partial_\mu v^{\alpha_1 \dots \alpha_s} \partial_{\lambda_2} \wedge \dots \wedge \partial_{\lambda_r} \wedge \partial_{\alpha_1} \wedge \dots \wedge \partial_{\alpha_s}). \end{aligned} \quad (4.18)$$

This generalizes the Lie bracket of vector fields. The relations

$$[\vartheta, v]_{\text{SN}} = (-1)^{|\vartheta||v|} [v, \vartheta]_{\text{SN}}, \quad (4.19)$$

$$[\nu, \vartheta \wedge v]_{\text{SN}} = [\nu, \vartheta]_{\text{SN}} \wedge v + (-1)^{(|\nu|-1)|\vartheta|} \vartheta \wedge [\nu, v]_{\text{SN}}, \quad (4.20)$$

$$\begin{aligned} (-1)^{|\nu|(|v|-1)} [\nu, [\vartheta, v]_{\text{SN}}]_{\text{SN}} &+ (-1)^{|\vartheta|(|\nu|-1)} [\vartheta, [\nu, v]_{\text{SN}}]_{\text{SN}} \\ &+ (-1)^{|\nu|(|\vartheta|-1)} [v, [\nu, \vartheta]_{\text{SN}}]_{\text{SN}} = 0, \end{aligned} \quad (4.21)$$

hold.

Example 4.1: Let

$$w = \frac{1}{2} w^{\mu\nu} \partial_\mu \wedge \partial_\nu \quad (4.22)$$

be a bivector field. Given the multivector field ϑ (4.17), the Schouten–Nijenhuis bracket $[w, \vartheta]_{\text{SN}}$ (4.18) reads

$$[w, \vartheta]_{\text{SN}} = \left[\frac{1}{r!} w^{\mu\alpha} \partial_\mu \vartheta^{\lambda\lambda_2 \dots \lambda_r} + \frac{1}{2(r-1)!} \vartheta^{\mu\lambda_2 \dots \lambda_r} \partial_\mu w^{\alpha\lambda} \right] \partial_\alpha \wedge \partial_{\lambda_1} \wedge \partial_{\lambda_2} \wedge \dots \wedge \partial_{\lambda_r}. \quad (4.23)$$

In particular,

$$[w, w]_{\text{SN}} = w^{\mu\lambda_1} \partial_\mu w^{\lambda_2 \lambda_3} \partial_{\lambda_1} \wedge \partial_{\lambda_2} \wedge \partial_{\lambda_3}. \quad (4.24)$$

□

A generalization of the interior product (4.16) to multivector fields is the left interior product

$$\vartheta \rfloor \phi = \phi(\vartheta), \quad |\vartheta| \leq |\phi|, \quad \phi \in \mathcal{O}^*(Z), \quad \vartheta \in \mathcal{T}_*(Z),$$

of multivector fields and exterior forms. It is defined by the equalities

$$\phi(u_1 \wedge \dots \wedge u_r) = \phi(u_1, \dots, u_r), \quad \phi \in \mathcal{O}^*(Z), \quad u_i \in \mathcal{T}_1(Z),$$

and obeys the relation

$$\vartheta \rfloor v \rfloor \phi = (v \wedge \vartheta) \rfloor \phi = (-1)^{|v||\vartheta|} v \rfloor \vartheta \rfloor \phi, \quad \phi \in \mathcal{O}^*(Z), \quad \vartheta, v \in \mathcal{T}_*(Z).$$

If $|\phi| \leq |\vartheta|$, the right interior product

$$\vartheta \rfloor \phi = \vartheta(\phi), \quad \phi \in \mathcal{O}^*(Z), \quad \vartheta \in \mathcal{T}_*(Z),$$

of exterior forms and multivector fields is given by the equalities

$$\begin{aligned} \vartheta(\phi_1, \dots, \phi_r) &= \vartheta \rfloor \phi_r \cdots \rfloor \phi_1, \quad \phi_i \in \mathcal{O}^1(Z), \quad \vartheta \in \mathcal{T}_r(Z), \\ \vartheta \rfloor \phi_i &= \frac{1}{(r-1)!} \vartheta^{\mu\alpha_2 \dots \alpha_r} \phi_{i\mu} \partial_{\alpha_2} \wedge \dots \wedge \partial_{\alpha_r}. \end{aligned} \quad (4.25)$$

It satisfies the relations

$$\begin{aligned} (\vartheta \wedge v) \rfloor \phi &= \vartheta \wedge (v \rfloor \phi) + (-1)^{|v|} (\vartheta \rfloor \phi) \wedge v, \quad \phi \in \mathcal{O}^1(Z), \\ \vartheta(\phi \wedge \sigma) &= \vartheta \rfloor \sigma \rfloor \phi, \quad \phi, \sigma \in \mathcal{O}^*(Z). \end{aligned}$$

A tangent-valued r -form on a manifold Z is a section

$$\phi = \frac{1}{r!} \phi_{\lambda_1 \dots \lambda_r}^\mu dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r} \otimes \partial_\mu \quad (4.26)$$

of the tensor bundle $\overset{r}{\wedge} T^*Z \otimes TZ \rightarrow Z$.

A (regular) distribution on a manifold Z is a subbundle \mathbf{T} of the tangent bundle TZ of Z . A vector field u on Z is said to be subordinate to a distribution \mathbf{T} if it lives in \mathbf{T} . A distribution \mathbf{T} is called involutive if the Lie bracket of \mathbf{T} -subordinate vector fields also is subordinate to \mathbf{T} . The following local coordinates can be associated to an involutive distribution [53].

THEOREM 4.4: Let \mathbf{T} be an involutive r -dimensional distribution on manifold Z , $\dim Z = k$. Every point $z \in Z$ has an open neighborhood U which is a domain of an adapted coordinate chart (z^1, \dots, z^k) such that, restricted to U , the distribution \mathbf{T} and its annihilator $\text{Ann } \mathbf{T}$ are spanned by the local vector fields $\partial/\partial z^1, \dots, \partial/\partial z^r$ and the one-forms dz^{r+1}, \dots, dz^k , respectively. \square

A connected submanifold N of a manifold Z is said to be an integral manifold of a distribution \mathbf{T} on Z if $TN \subset \mathbf{T}$. Unless otherwise stated, by an integral manifold is meant an integral manifold of dimension of \mathbf{T} . An integral manifold is called maximal if no other integral manifold contains it. The following is the classical theorem of Frobenius [26, 53].

THEOREM 4.5: Let \mathbf{T} be an involutive distribution on a manifold Z . Through any point $z \in Z$, there passes a unique maximal integral manifold of \mathbf{T} , and any integral manifold through z is its open subset. \square

Maximal integral manifolds of an involutive distribution on a manifold Z are assembled into a (regular) foliation \mathcal{F} of Z . It is defined as a partition of Z into connected r -dimensional submanifolds (leaves of a foliation) F_ι , $\iota \in I$, which possesses the following properties [44, 49]. A foliated manifold (Z, \mathcal{F}) admits an adapted coordinate atlas

$$\{(U_\xi; z^\lambda; z^i)\}, \quad \lambda = 1, \dots, n-r, \quad i = 1, \dots, r, \quad (4.27)$$

such that transition functions of coordinates z^λ are independent of the remaining coordinates z^i and, for each leaf F of a foliation \mathcal{F} , the connected components of $F \cap U_\xi$ are given by the equations $z^\lambda = \text{const}$. These connected components and coordinates (z^i) on them make up a coordinate atlas of a leaf F .

4.3 First order jet manifolds

Given a fibre bundle $Y \rightarrow X$ with bundle coordinates (x^λ, y^i) , let us consider the equivalence classes $j_x^1 s$ of its sections s identified by their values $s^i(x)$ and values of their derivatives $\partial_\mu s^i(x)$ at points $x \in X$. They are called the first order jets of sections. The particular choice of coordinates does not matter for this definition. The key point is that the set $J^1 Y$ of first order jets $j_x^1 s$, $x \in X$, is a smooth manifold with respect to the adapted coordinates $(x^\lambda, y^i, y_\lambda^i)$ such that

$$y_\lambda^i(j_x^1 s) = \partial_\lambda s^i(x), \quad y_\lambda^i = \frac{\partial x^\mu}{\partial x'^\lambda} (\partial_\mu + y_\mu^j \partial_j) y^i. \quad (4.28)$$

It is called the first order jet manifold of a fibre bundle $Y \rightarrow X$ [19, 47, 48].

The jet manifold J^1Y admits the natural fibrations

$$\pi^1 : J^1Y \ni j_x^1 s \rightarrow x \in X, \quad \pi_0^1 : J^1Y \ni j_x^1 s \rightarrow s(x) \in Y, \quad (4.29)$$

where the second one is an affine bundle.

There are the canonical imbeddings

$$\lambda_1 : J^1Y \xrightarrow{Y} T^*X \otimes_Y TY, \quad \lambda_1 = dx^\lambda \otimes (\partial_\lambda + y_\lambda^i \partial_i) = dx^\lambda \otimes d_\lambda, \quad (4.30)$$

$$\theta_1 : J^1Y \xrightarrow{Y} T^*Y \otimes_Y VY, \quad \theta_1 = (dy^i - y_\lambda^i dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \quad (4.31)$$

where d_λ are total derivatives and θ^i are called contact forms. Identifying the jet manifold J^1Y to its images under the canonical morphisms (4.30) and (4.31), one can represent jets $j_x^1 s = (x^\lambda, y^i, y_\mu^i)$ by tangent-valued forms

$$dx^\lambda \otimes (\partial_\lambda + y_\lambda^i \partial_i), \quad (dy^i - y_\lambda^i dx^\lambda) \otimes \partial_i. \quad (4.32)$$

Sections and morphisms of fibre bundles admit prolongations to jet manifolds as follows.

- Every section s of a fibre bundle $Y \rightarrow X$ has the jet prolongation to the section

$$(J^1 s)(x) := j_x^1 s, \quad y_\lambda^i \circ J^1 s = \partial_\lambda s^i(x),$$

of the jet bundle $J^1Y \rightarrow X$. A section \bar{s} of the jet bundle $J^1Y \rightarrow X$ is called holonomic or integrable if it is the jet prolongation of some section of the fibre bundle $Y \rightarrow X$.

- Every bundle morphism $\Phi : Y \rightarrow Y'$ over a diffeomorphism f admits a jet prolongation to the bundle morphism over Φ of the affine jet bundles

$$J^1 \Phi : J^1 Y \xrightarrow{\Phi} J^1 Y', \quad y_\lambda^i \circ J^1 \Phi = \frac{\partial (f^{-1})^\mu}{\partial x'^\lambda} d_\mu \Phi^i.$$

- Every projectable vector field u on a fibre bundle $Y \rightarrow X$ has a jet prolongation to the projectable vector field

$$J^1 u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda \quad (4.33)$$

on the affine jet bundle $J^1Y \rightarrow Y$.

4.4 Higher and infinite order jets

The notion of a first order jet manifolds is naturally extended to higher order jets.

Let $Y \rightarrow X$ be a fibre bundle. Given its bundle coordinates (x^λ, y^i) , a multi-index Λ of the length $|\Lambda| = r$ throughout denotes a collection of numbers $(\lambda_r \dots \lambda_1)$ modulo permutations. By $\Lambda + \Sigma$ is meant a multi-index $(\lambda_r \dots \lambda_1 \sigma_k \dots \sigma_1)$. Let use the notation

$$\partial_\Lambda = \partial_{\lambda_r} \circ \dots \circ \partial_{\lambda_1}. \quad (4.34)$$

The r -order jet manifold $J^r Y$ of sections of a fibre bundle $Y \rightarrow X$ (or, simply, the jet manifold of $Y \rightarrow X$) is defined as the disjoint union of the equivalence classes $j_x^r s$ of sections s of Y such that different sections s and s' belong to the same equivalence class $j_x^r s$ iff

$$s^i(x) = s'^i(x), \quad \partial_\Lambda s^i(x) = \partial_\Lambda s'^i(x), \quad 0 < |\Lambda| \leq r.$$

In brief, one can say that sections of $Y \rightarrow X$ are identified by the $r+1$ terms of their Taylor series at points of X . The particular choice of coordinates does not matter for this definition. Given bundle coordinates (x^λ, y^i) on a fibre bundle $Y \rightarrow X$, the set $J^r Y$ is endowed with an atlas of the adapted coordinates

$$(x^\lambda, y_\Lambda^i), \quad y_\Lambda^i \circ s = \partial_\Lambda s^i(x), \quad 0 \leq |\Lambda| \leq r, \quad (4.35)$$

$$y'_{\lambda+\Lambda}^i = \frac{\partial x^\mu}{\partial' x^\lambda} d_\mu y_\Lambda^i, \quad (4.36)$$

where the symbol d_λ stands for the total derivative

$$d_\lambda = \partial_\lambda + \sum_{|\Lambda|=0}^{r-1} y_{\Lambda+\lambda}^i \partial_i^\Lambda. \quad (4.37)$$

Let us also use the notation

$$d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}. \quad (4.38)$$

The coordinates (4.35) bring the set $J^r Y$ into a smooth manifold. They are compatible with the natural surjections

$$\pi_l^r : J^r Y \rightarrow J^l Y, \quad r > l,$$

which form the composite bundle

$$\begin{aligned} \pi^r : J^r Y &\xrightarrow{\pi_{r-1}^r} J^{r-1} Y \xrightarrow{\pi_{r-2}^{r-1}} \cdots \xrightarrow{\pi_0^1} Y \xrightarrow{\pi} X, \\ \pi_h^k \circ \pi_k^r &= \pi_h^r, \quad \pi^h \circ \pi_h^r = \pi^r. \end{aligned}$$

A glance at the transition functions (4.36) when $|\Lambda| = r$ shows that the fibration $J^r Y \rightarrow J^{r-1} Y$ is an affine bundle.

Remark 4.2: In order to introduce higher order jet manifolds, one can use the construction of the repeated jet manifolds. Let us consider the r -order jet manifold $J^r J^k Y$ of the jet bundle $J^k Y \rightarrow X$. It is coordinated by $(x^\mu, y_{\Sigma\Lambda}^i)$, $|\Lambda| \leq k$, $|\Sigma| \leq r$. There is the canonical monomorphism

$$\sigma_{rk} : J^{r+k} Y \rightarrow J^r J^k Y, \quad y_{\Sigma\Lambda}^i \circ \sigma_{rk} = y_{\Sigma+\Lambda}^i. \quad (4.39)$$

□

In the calculus in higher order jets, we have the r -order jet prolongation functor such that, given fibre bundles Y and Y' over X , every bundle morphism

$\Phi : Y \rightarrow Y'$ over a diffeomorphism f of X admits the r -order jet prolongation to a morphism of r -order jet manifolds

$$J^r \Phi : J^r Y \ni j_x^r s \rightarrow j_{f(x)}^r (\Phi \circ s \circ f^{-1}) \in J^r Y'. \quad (4.40)$$

The jet prolongation functor is exact. If Φ is an injection (resp. a surjection), so is $J^r \Phi$. It also preserves an algebraic structure. In particular, if $Y \rightarrow X$ is a vector bundle, so is $J^r Y \rightarrow X$.

Every section s of a fibre bundle $Y \rightarrow X$ admits the r -order jet prolongation to the section

$$(J^r s)(x) = j_x^r s, \quad (4.41)$$

called an integrable section, of the jet bundle $J^r Y \rightarrow X$. Thus, the jet prolongation functor yields an injection

$$Y(X) \ni s \rightarrow J^r s \in J^r Y(X) \quad (4.42)$$

of a set $Y(X)$ of global sections of a fibre bundle $Y \rightarrow X$ onto a subset of integrable sections (4.41) of a set of global sections $J^r Y(X)$ of the jet bundle $J^r Y \rightarrow X$.

Every exterior form ϕ on the jet manifold $J^k Y$ gives rise to the pull-back form $\pi_k^{k+i*} \phi$ on the jet manifold $J^{k+i} Y$. Let $\mathcal{O}_k^* = \mathcal{O}^*(J^k Y)$ denote the algebra of exterior forms on the jet manifold $J^k Y$. We have the direct system of differential graded algebras

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi_0^*} \mathcal{O}_1^* \xrightarrow{\pi_1^*} \dots \xrightarrow{\pi_{r-1}^*} \mathcal{O}_r^* \longrightarrow \dots \quad (4.43)$$

Higher order jet formalism provides the adequate formulation of theory of (non-linear) differential operators [5, 19, 31]. Let $E \rightarrow X$ be a fibre bundle coordinated by (x^λ, v^A) , $A = 1, \dots, m$.

DEFINITION 4.1: A k -order E -valued differential operator on a fibre bundle $Y \rightarrow X$ is defined as a section Δ of the pull-back bundle

$$J^k Y \times_X E \rightarrow J^k Y. \quad (4.44)$$

□

There is obvious one-to-one correspondence between the sections Δ of the fibre bundle (4.44) and the bundle morphisms

$$\Delta : J^k Y \xrightarrow{\quad} E, \quad v^A \circ \Delta = \Delta^A(x^\lambda, y^i, y_\lambda^i, \dots, y_{\lambda_k \dots \lambda_1}^i). \quad (4.45)$$

Such a morphism also is called a k -order differential operator on a fibre bundle $Y \rightarrow X$. It sends each section s of $Y \rightarrow X$ onto the section

$$(\Delta \circ J^k s)^A(x) = \Delta^A(x^\lambda, s^i(x), \partial_\lambda s^i(x), \dots, \partial_{\lambda_k} \dots \partial_{\lambda_1} s^i(x)) \quad (4.46)$$

of a fibre bundle $E \rightarrow X$. Therefore, there is the following equivalent definition of differential operators on Y .

DEFINITION 4.2: Let $Y \rightarrow X$ and $E \rightarrow X$ be fibre bundles. The bundle morphism $J^k Y \rightarrow E$ (4.45) over X is called the E -valued k -order differential operator on $Y \rightarrow X$. \square

Finite order jet manifolds make up the inverse system

$$X \xleftarrow{\pi} Y \xleftarrow{\pi_0^1} \dots \xleftarrow{\pi_{r-1}^r} J^r Y \xleftarrow{\pi_r^{r+1}} J^{r+1} Y \xleftarrow{\pi_{r+1}^{r+2}} \dots. \quad (4.47)$$

Its projective limit

$$J^\infty Y = \varprojlim J^r Y, \quad (4.48)$$

is defined as a set $J^\infty Y$, whose elements represent ∞ -jets $j_x^\infty s$ of local sections of $Y \rightarrow X$, together with surjections

$$\pi^\infty : J^\infty Y \rightarrow X, \quad \pi_0^\infty : J^\infty Y \rightarrow Y, \quad \pi_k^\infty : J^\infty Y \rightarrow J^k Y, \quad (4.49)$$

such that $\pi_r^\infty = \pi_r^k \circ \pi_k^\infty$ for any admissible k and $r < k$. Sections of Y belong to the same jet $j_x^\infty s$ iff their Taylor series at a point $x \in X$ coincide with each other. Therefore, $J^\infty Y$ is called the infinite order jet space.

The set $J^\infty Y$ is provided with the inverse limit topology. This is the coarsest topology such that the surjections (4.49) are continuous. The base of open sets of this topology in $J^\infty Y$ consists of the inverse images of open subsets of $J^k Y$, $k = 0, \dots$, under the mappings (4.49). This topology makes $J^\infty Y$ into a paracompact Fréchet manifold. A bundle coordinate atlas $\{U, (x^\lambda, y^i)\}$ of $Y \rightarrow X$ yields the manifold coordinate atlas

$$\{(\pi_0^\infty)^{-1}(U_Y), (x^\lambda, y_\Lambda^i)\}, \quad 0 \leq |\Lambda|, \quad (4.50)$$

of $J^\infty Y$, together with the transition functions (4.36) where d_λ denotes the total derivative

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y_{\lambda+\Lambda}^i \partial_i^\Lambda. \quad (4.51)$$

Though $J^\infty Y$ fails to be a smooth manifold, one can introduce the differential calculus on $J^\infty Y$ as follows.

Let us consider the direct system (4.43) of \mathbb{R} -modules \mathcal{O}_k^* of exterior forms on finite order jet manifolds $J^k Y$. Its direct limit

$$\mathcal{O}_\infty^* = \varinjlim \mathcal{O}_k^* \quad (4.52)$$

together with monomorphisms

$$\pi_r^{\infty*} : \mathcal{O}_k^* \rightarrow \mathcal{O}_\infty^*$$

is an $C^\infty(X)$ -module which consists of all the exterior forms on finite order jet manifolds module pull-back identification. The operations of the exterior

product \wedge and the exterior differential d also have the direct limits on \mathcal{O}_∞^* . They provide \mathcal{O}_∞^* with the structure of the differential graded algebra

$$0 \rightarrow \mathbb{R} \longrightarrow \mathcal{O}_\infty^0 \xrightarrow{d} \mathcal{O}_\infty^1 \xrightarrow{d} \dots, \quad (4.53)$$

where \mathcal{O}_∞^m are the direct limits of the direct systems

$$\mathcal{O}^m(X) \xrightarrow{\pi^*} \mathcal{O}_0^m \xrightarrow{\pi_0^{1*}} \mathcal{O}_1^m \longrightarrow \dots \mathcal{O}_r^m \xrightarrow{\pi_r^{r+1*}} \mathcal{O}_{r+1}^m \longrightarrow \dots \quad (4.54)$$

of real vector spaces \mathcal{O}_r^m of exterior m -forms on r -order jet manifolds $J^r Y$. We agree to call elements of \mathcal{O}_∞^* the exterior forms on the infinite order jet space $J^\infty Y$.

Given a manifold coordinate atlas (4.50) of $J^\infty Y$, elements of the direct limit \mathcal{O}_∞^* can be written in the coordinate form as exterior forms on finite order jet manifolds. The basic one-forms dx^λ and the contact forms $\theta_\Lambda^i = dy_\Lambda^i - y_{\Lambda+\Lambda}^i dx^\lambda$ make up the set of local generators of the \mathcal{O}_∞^0 -algebra \mathcal{O}_∞^* .

Accordingly, the exterior differential on \mathcal{O}_∞^* is split into the sum $d = d_H + d_V$ of the horizontal (total) and vertical differentials such that

$$\begin{aligned} d_H(\phi) &= dx^\lambda \wedge d_\lambda(\phi), & d_V(\phi) &= \theta_\Lambda^i \wedge \partial_i^\Lambda \phi, & \phi &\in \mathcal{O}_\infty^*, \\ d_H \circ d_H &= 0, & d_V \circ d_V &= 0, & d_V \circ d_H + d_H \circ d_V &= 0. \end{aligned} \quad (4.55)$$

4.5 Hochschild cohomology

This and next Sections summarize the relevant basics on homology and cohomology of algebraic systems [15, 20, 34, 38].

Let \mathcal{K} be a commutative ring and \mathcal{A} a \mathcal{K} -ring which need not be commutative. Hochschild cohomology is the cohomology of a \mathcal{K} -ring \mathcal{A} with coefficients in a \mathcal{A} -bimodule. Throughout this Section, by \otimes is meant the tensor product of modules over \mathcal{K} .

Let $B_k(\mathcal{A})$, $k \in \mathbb{N}_+$, be an \mathcal{A} -bimodule whose basis is the tensor product of \mathcal{K} -modules $\otimes_k \mathcal{A}$, i.e., it consists of elements $[a_1 \otimes \dots \otimes a_k]$, $a_i \in \mathcal{A}$. The $B_0(\mathcal{A})$ is defined as an \mathcal{A} -bimodule of rank 1 whose basis element is denoted by $[]$. One considers the chain complex $B_*(\mathcal{A})$ with respect to boundary operators defined as the \mathcal{A} -bimodule morphisms

$$\begin{aligned} \partial_{k>0}[a_1 \otimes \dots \otimes a_k] &= a_1[a_2 \otimes \dots \otimes a_k] + \\ &\sum_{j=1}^{k-1} (-1)^j [a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_k] + (-1)^k [a_1 \otimes \dots \otimes a_{k-1}] a_k. \end{aligned} \quad (4.56)$$

In particular, we have

$$\partial_0 : B_0(\mathcal{A}) \ni a[] a' \rightarrow aa' \in \mathcal{A}, \quad \partial_1(a[a_1]a') = aa_1[] a' - a[] a_1 a'.$$

This chain complex admits the homotopy operator given by the right \mathcal{A} -module morphisms

$$\mathbf{h} : a[a_1 \otimes \dots \otimes a_k] \rightarrow [a \otimes a_1 \otimes \dots \otimes a_k], \quad k \in \mathbb{N}. \quad (4.57)$$

For instance, we have $\mathbf{h}_0(a[]a') = [a]a'$. It follows that the chain complex $B_*(\mathcal{A})$ is acyclic. The corresponding exact sequence reads

$$0 \longleftarrow \mathcal{A} \xleftarrow{\partial_0} B_0(\mathcal{A}) \longleftarrow \cdots B_k(\mathcal{A}) \xleftarrow{\partial_{k+1}} \cdots . \quad (4.58)$$

It is readily observed that the homotopy operator \mathbf{h} (4.57), completed by the map

$$h : \mathcal{A} \ni a \rightarrow []a \in B_0(\mathcal{A}),$$

is the homotopy operator for the chain complex (4.58). Therefore, this chain complex is exact at all terms $B_k(\mathcal{A})$, $k \in \mathbb{N}$. Moreover, it is exact at \mathcal{A} since \mathcal{A} has a unit element and, therefore, ∂_0 is a \mathcal{K} -module epimorphism. Hence, the chain complex (4.58) is a resolution of the ring \mathcal{A} by \mathcal{A} -bimodules.

Remark 4.3: Given the canonical monomorphism of $\mathcal{K} \rightarrow \mathcal{A}$, one also constructs the chain complex of \mathcal{A} -bimodules

$$C_0(\mathcal{A}) = B_0(\mathcal{A}), \quad C_{k>0}(\mathcal{A}) = \bigotimes^k (\mathcal{A}/\mathcal{K}) \quad (4.59)$$

whose boundary operators take the form (4.56). The chain complexes $B_*(\mathcal{A})$ and $C_*(\mathcal{A})$ are proved to be homotopic [34]. \square

Let us turn now to Hochschild cohomology. Let Q be an \mathcal{A} -bimodule. Given the chain complex $B_*(\mathcal{A})$, let us consider the cochain complex

$$0 \rightarrow B^0(\mathcal{A}, Q) \xrightarrow{\delta^0} B^1(\mathcal{A}, Q) \rightarrow \cdots B^k(\mathcal{A}, Q) \xrightarrow{\delta^k} \cdots , \quad (4.60)$$

whose terms are the \mathcal{A} -bimodules

$$B^k(\mathcal{A}, Q) = \text{Hom}_{\mathcal{K}}(B_k(\mathcal{A}), Q).$$

It is called the Hochschild complex. Their elements can be seen as Q -valued \mathcal{K} -multilinear functions $f^k(a_1, \dots, a_k)$ on \mathcal{A} . The coboundary operators read

$$\begin{aligned} (\delta^k f^k)(a_1, \dots, a_{k+1}) &= f^k(\partial_{k+1}[a_1 \otimes \cdots \otimes a_{k+1}]) = \\ &= a_1 f^k(a_2, \dots, a_{k+1}) + \sum_j (-1)^j f^k(a_1, \dots, a_j a_{j+1}, \dots, a_{k+1}) \\ &+ (-1)^{k+1} f^k(a_1, \dots, a_k) a_{k+1}, \quad k \in \mathbb{N}. \end{aligned} \quad (4.61)$$

In particular, the module $B^0(\mathcal{A}, Q)$ is isomorphic to Q via the association

$$Q \ni q \rightarrow f_q^0 \in B^0(\mathcal{A}, Q), \quad f_q^0([]) = q.$$

For instance, we have

$$\delta^0 f_q^0(a) = aq - qa, \quad a \in \mathcal{A}, \quad (4.62)$$

$$\delta^1 f^1(a_1, a_2) = a_1 f^1(a_2) - f^1(a_1 a_2) + f^1(a_1) a_2, \quad (4.63)$$

$$\begin{aligned} \delta^2 f^2(a_1, a_2, a_3) &= a_1 f^2(a_2, a_3) - f^2(a_1 a_2, a_3) + \\ &+ f^2(a_1, a_2 a_3) - f^2(a_1, a_2) a_3, \quad a_1, a_2, a_3 \in \mathcal{A}. \end{aligned} \quad (4.64)$$

DEFINITION 4.3: Cohomology $H^*(\mathcal{A}, Q)$ of the complex $B^*(\mathcal{A}, Q)$ (4.60) is called the Hochschild cohomology of a \mathcal{K} -ring \mathcal{A} with coefficients in an \mathcal{A} -module Q . \square

The Hochschild complex $B^*(\mathcal{A}, Q)$ (4.60) is functorial in Q , i.e., every \mathcal{A} -bimodule morphism $Q \rightarrow P$ yields a cochain \mathcal{K} -module morphism $B^*(\mathcal{A}, Q) \rightarrow B^*(\mathcal{A}, P)$ and, consequently, a homomorphism of Hochschild cohomology groups $H^*(\mathcal{A}, Q) \rightarrow H^*(\mathcal{A}, P)$ [17]. In particular, an \mathcal{A} -bimodule morphism $\Phi : Q \rightarrow P$ is called \mathcal{K} -allowable if there exists a \mathcal{K} -module morphism $\lambda : P \rightarrow Q$ such that $\Phi \circ \lambda \circ \Phi = \Phi$. This condition is always satisfied if \mathcal{K} is a field.

The Hochschild cohomology modules of low degrees have the following natural interpretation.

(i) Since $H^0(\mathcal{A}, Q) = \text{Ker } \delta^0 = Q$, it follows from the expression (4.62) that $H^0(\mathcal{A}, Q)$ is isomorphic to the center \mathcal{Z}_Q of the \mathcal{A} -bimodule Q .

(ii) A glance at the expression (4.63) shows that any normalized one-cocycle f^1 is a Q -valued derivation of the \mathcal{K} -ring \mathcal{A} and *vice versa*. In particular, one-coboundaries $\delta^0 f_q^0$ are inner Q -valued derivations up to the sign minus. Therefore, $H^1(\mathcal{A}, Q)$ is sometimes called the module of outer derivations of \mathcal{A} .

In particular, let Q be \mathcal{A} itself seen as an \mathcal{A} -bimodule. In this case, the Hochschild complex $B^*(\mathcal{A}, \mathcal{A})$ is provided with the following two products [17].

(i) The first one is the cup-product

$$f \smile f'(a_1, \dots, a_{m+n}) = f(a_1, \dots, a_m) f'(a_{m+1}, \dots, a_{m+n}), \quad (4.65)$$

which obeys the equality

$$\delta^{m+n}(f \smile f') = \delta^m f \smile f' + (-1)^m f \smile \delta^n f'.$$

Hence, it induces the cup-product

$$[f] \smile [f'] = [f \smile f'] \quad (4.66)$$

of the Hochschild cohomology classes $[f], [f'] \in H^*(\mathcal{A}, \mathcal{A})$. This product makes $H^*(\mathcal{A}, \mathcal{A})$ into a graded commutative algebra, i.e., the relation

$$[f] \smile [f'] = (-1)^{|f||f'|} [f'] \smile [f]$$

holds, where $|f|$ denotes the degree of cohomology classes.

(ii) The composition product of $f \in B^m(\mathcal{A}, \mathcal{A})$ and $f' \in B^n(\mathcal{A}, \mathcal{A})$ is defined by the formula

$$f \circ f'(a_1, \dots, a_{m+n-1}) = \sum_{i=1}^m (-1)^{(i-1)(n-1)} f(a_1, \dots, a_{i-1}, f'(a_i, \dots, a_{n+i-1}), a_{n+i}, \dots, a_{m+n-1}). \quad (4.67)$$

It obeys the relation

$$\delta(f \circ f') = (-1)^{n-1} \delta f \circ f' + f \circ \delta f' + (-1)^n (f' \smile f - (-1)^{mn} f \smile f').$$

It is easily justified that, if f and f' are cocycles, the bracket

$$[f, f']^\circ = f \circ f' - (-1)^{(m-1)(n-1)} f' \circ f \quad (4.68)$$

is also a cocycle. Consequently, the bracket (4.68) induces the bracket of the Hochschild cohomology classes

$$[[f], [f']]^\circ = [[f, f']^\circ], \quad (4.69)$$

called the Gerstenhaber bracket. This bracket makes $H^*(\mathcal{A}, \mathcal{A})$ into a graded Lie algebra where elements $[f] \in H^m(\mathcal{A}, \mathcal{A})$ are provided with the graded degree $|f| - 1 = m - 1$.

The cup-product (4.66) and the Gerstenhaber bracket (4.69) in the Hochschild cohomology $H^*(\mathcal{A}, \mathcal{A})$ satisfy the relations making $H^*(\mathcal{A}, \mathcal{A})$ into a Gerstenhaber algebra.

4.6 Chevalley–Eilenberg cohomology of Lie algebras

One associates the Chevalley–Eilenberg complex to a Lie algebra.

In this Section, \mathcal{G} denotes a Lie algebra (not necessarily finite-dimensional) over a commutative ring \mathcal{K} .

Let \mathcal{G} act on a \mathcal{K} -module P on the left by endomorphisms

$$\begin{aligned} \mathcal{G} \times P &\ni (\varepsilon, p) \rightarrow \varepsilon p \in P, \\ [\varepsilon, \varepsilon'] p &= (\varepsilon \circ \varepsilon' - \varepsilon' \circ \varepsilon) p, \quad \varepsilon, \varepsilon' \in \mathcal{G}. \end{aligned}$$

One says that P is a \mathcal{G} -module. A \mathcal{K} -multilinear skew-symmetric map

$$c^k : \times^k \mathcal{G} \rightarrow P$$

is called a P -valued k -cochain on the Lie algebra \mathcal{G} . These cochains form a \mathcal{G} -module $C^k[\mathcal{G}; P]$. Let us put $C^0[\mathcal{G}; P] = P$. We obtain the cochain complex

$$0 \rightarrow P \xrightarrow{\delta^0} C^1[\mathcal{G}; P] \xrightarrow{\delta^1} \dots C^k[\mathcal{G}; P] \xrightarrow{\delta^k} \dots, \quad (4.70)$$

with respect to the Chevalley–Eilenberg coboundary operators

$$\begin{aligned} \delta^k c^k(\varepsilon_0, \dots, \varepsilon_k) &= \sum_{i=0}^k (-1)^i \varepsilon_i c^k(\varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \varepsilon_k) + \\ &\quad \sum_{1 \leq i < j \leq k} (-1)^{i+j} c^k([\varepsilon_i, \varepsilon_j], \varepsilon_0, \dots, \widehat{\varepsilon}_i, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_k), \end{aligned} \quad (4.71)$$

where the caret $\widehat{}$ denotes omission [15]. The complex (4.70) is called the Chevalley–Eilenberg complex with coefficients in a module P . It is finite if the Lie algebra \mathcal{G} is finite-dimensional.

For instance,

$$\delta^0 p(\varepsilon_0) = \varepsilon_0 p, \quad (4.72)$$

$$\delta^1 c^1(\varepsilon_0, \varepsilon_1) = \varepsilon_0 c^1(\varepsilon_1) - \varepsilon_1 c^1(\varepsilon_0) - c^1([\varepsilon_0, \varepsilon_1]). \quad (4.73)$$

Cohomology $H^*(\mathcal{G}; P)$ of the complex $C^*[\mathcal{G}; P]$ is called the Chevalley–Eilenberg cohomology of the Lie algebra \mathcal{G} with coefficients in the module P .

In particular, let $P = \mathcal{G}$ be regarded as a \mathcal{G} -module with respect to the adjoint representation

$$\varepsilon : \varepsilon' \rightarrow [\varepsilon, \varepsilon'] \quad , \varepsilon, \varepsilon' \in \mathcal{G}.$$

We abbreviate with $C^*[\mathcal{G}]$ the Chevalley–Eilenberg complex of \mathcal{G} -valued cochains on \mathcal{G} .

DEFINITION 4.4: Cohomology $H^*(\mathcal{G})$ of this complex is called the Chevalley–Eilenberg cohomology or, simply, the cohomology of a Lie algebra \mathcal{G} . \square

In particular, $C^0[\mathcal{G}] = \mathcal{G}$, while $C^1[\mathcal{G}]$ consists of endomorphisms of the Lie algebra \mathcal{G} . Accordingly, the coboundary operators (4.72) and (4.73) read

$$\delta^0 \varepsilon(\varepsilon_0) = [\varepsilon_0, \varepsilon], \quad (4.74)$$

$$\delta^1 c^1(\varepsilon_0, \varepsilon_1) = [\varepsilon_0, c^1(\varepsilon_1)] - [\varepsilon_1, c^1(\varepsilon_0)] - c^1([\varepsilon_0, \varepsilon_1]). \quad (4.75)$$

A glance at the expression (4.75) shows that a one-cocycle c^1 on \mathcal{G} obeys the relation

$$c^1([\varepsilon_0, \varepsilon_1]) = [c^1(\varepsilon_0), \varepsilon_1] + [\varepsilon_0, c^1(\varepsilon_1)]$$

and, thus, it is a derivation of the Lie algebra \mathcal{G} . Accordingly, any one-coboundary (4.74) is an inner derivation of \mathcal{G} up to the sign minus. Therefore, one can think of the cohomology $H^1(\mathcal{G})$ as being the set of outer derivations of \mathcal{G} . One can show that the cohomology $H^2(\mathcal{G})$ indexes non-equivalent infinitesimal deformations of the Lie algebra \mathcal{G} [15].

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